Integral geometry of spherically convex bodies[1]

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1 Subject of the dissertation

A recent survey article on integral geometry of convex bodies in Euclidean space was written by Schneider & Wieacker [10], introductory accounts with full proofs are Schneider & Weil [9] and section 4.5 in Schneider [7].

The above authors' approach utilizes results about convex bodies and invariant measures on homogeneous spaces, in particular no methods from differential geometry are required. The main results are local kinematic formulas for convex bodies and finite unions of them.

The aim of the dissertation [4] was to investigate whether and to what extend this approach can be pursued also in spherical space. This question is interesting because the sphere S^n and its motion group SO_{n+1} lack the product structure of Euclidean space and its motion group, which is essential in some integral geometric proofs. Also the principle of spherical duality suggests some additional questions, which can also be considered in the Euclidean case, but have not been treated up to now.

In the following we summarize our results. The main theorems are stated in sections 5 to 7, while sections 3 and 4 contain auxiliary material.

2 Notations

We use the following notations:

[1]AMS subject classifications (1991): 52A22, 52A55. This is a summary of the doctoral thesis [4].

$\mathbb{R}^{n+1}$	Euclidean $(n+1)$-space, equipped with standard inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$,
S^n	unit sphere of $\mathbb{R}^{n+1}$ (set of all unit vectors),
$\mathcal{K}$	set of all (spherically) convex bodies, i.e. all compact subsets $K \subset S^n$ such that for all $x, y \in K$ with $x \neq -y$ the spherical segment joining x and y is contained in K,
$\mathcal{R}$	set of all finite unions of elements of $\mathcal{K}$,
$K \vee K'$	spherically convex hull of $K, K' \in \mathcal{K}$, i.e. the union of all spherical segments joining all $x, y \in K \cup K'$ with $x \neq -y$,
$\mathcal{P}$	set of all (spherically convex) polytopes, i.e. all spherically convex hulls of finitely many one-point sets,
$\mathcal{F}(P)$	set of all faces of a polytope $P \in \mathcal{P}$ (including $\emptyset$ and P),
$\mathcal{S}_j$	set of all j-subspheres, i.e. all sets $L \cap S^n$ with $(j+1)$-dimensional linear subspaces L of $\mathbb{R}^{n+1}$,
$\mathcal{B}(X)$	σ-algebra of all Borel subsets of a topological space X,
β_i	$:= 2\pi^{(i+1)/2}/\Gamma(\frac{i+1}{2})$, the surface area of the i-dimensional unit sphere,
SO_{n+1}	group of all proper rotations of $\mathbb{R}^{n+1}$,
ν	invariant probability measure on $\mathcal{B}(SO_{n+1})$,
ν_j	image measure of ν under the map $SO_{n+1} \to \mathcal{S}_j$, $\rho \mapsto \rho S$ (where $S \in \mathcal{S}_j$ is chosen arbitrarily),
λ^n	spherical Lebesgue measure on $\mathcal{B}(S^n)$, normalized by $\lambda^n(S^n) = \beta_n$,
$d(x,y)$	$:= \arccos\langle x, y\rangle$, spherical distance of $x, y \in S^n$,
$d(K,x)$	$:= \min\{d(x,y) : y \in K\}$, distance of a point $x \in S^n$ from $K \in \mathcal{K}$ (in case $K = \emptyset$ we let $d(K,x) := \pi/2$ for all $x \in S^n$),
K_ϵ	$:= \{x \in S^n : d(K,x) \leq \epsilon\}$, parallel body of $K \in \mathcal{K}$,
K^*	$:= \{x \in S^n : \langle x, y\rangle \leq 0 \text{ for all } y \in K\}$, polar body of $K \in \mathcal{K}$,
$p(K,x)$	the unique point in $K \in \mathcal{K}$ with smallest distance to $x \in S^n$, where $d(K,x) < \pi/2$,
$u(K,x)$	$:= p(K^*, x)$ for $K \in \mathcal{K}$, $x \in S^n \backslash K$ with $d(K,x) < \pi/2$,
Nor K	$:= \{(x,u) \in K \times K^* : \langle x, u\rangle = 0\}$, set of all support elements of $K \in \mathcal{K}$.

Equipped with the Hausdorff metric δ defined by $\delta(K,K') := \min\{\epsilon \geq 0 : K \subset K'_\epsilon$ and $K' \subset K_\epsilon\}$, $\mathcal{K}$ becomes a compact metric space. The subset $\mathcal{P}$ is dense in $\mathcal{K}$.

3 Steiner formulae

From our standpoint the basic functionals of spherically convex bodies are the support measures, which are defined within the following theorem.

Theorem 1. *For $K \in \mathcal{K}$ there exist uniquely determined finite Borel measures $\Theta_j(K,\cdot)$, $j \in \{0,\dots,n-1\}$, on $S^n \times S^n$ such that for $0 < \epsilon < \pi/2$ and $\eta \in \mathcal{B}(S^n \times S^n)$ we have*

$$\lambda^n(\{x \in S^n : 0 < d(K,x) \leq \epsilon, (p(K,x), u(K,x)) \in \eta\})$$

$$= \sum_{j=0}^{n-1} \beta_j \beta_{n-j-1} \Theta_j(K,\eta) \int_0^{\epsilon} \cos^j t \sin^{n-j-1} t \, dt \, .$$

The measure $\Theta_j(K,\cdot)$ defined by this local Steiner formula is called the jth generalized curvature measure or the jth support measure of $K \in \mathcal{K}$. A very general Steiner-type result in space forms was proved by Kohlmann [5] using deeper methods from differential geometry and geometric measure theory. We proved Theorem 1 in an elementary way by first computing the local parallel volume in the case of polytopes and then applying an approximation procedure. The measures $\Theta_j(K,\cdot)$ are weakly continuous in $K \in \mathcal{K}$. If K is a polytope or a sufficiently smooth convex body, there are direct geometric interpretations of the measures $\Theta_j(K,\cdot)$, see [4], pp. 48 - 50, where one can also find explicit representations for the cases $j = n-1$ and $j = 0$ for general $K \in \mathcal{K}$.

By specialization we get some further functionals from the support measures. The measures defined on $\mathcal{B}(S^n)$ by

$$\Phi_j(K,A) := \Theta_j(K, A \times S^n), \quad j \in \{0,\dots,n-1\}, \qquad \Phi_n(K,A) := \frac{1}{\beta_n}\lambda^n(K \cap A)$$

are called curvature measures, the global measures

$$V_j(K) := \Phi_j(K, S^n), \qquad j \in \{0,\dots,n\},$$

we call the intrinsic volumes and the vectors

$$k_j(K) := \int_{S^n} x \, d\Phi_j(K,x) \ \in \mathbb{R}^{n+1}$$

the curvature vectors of $K \in \mathcal{K}$. The functionals $K \mapsto \Theta_j(K,\cdot)$ can be extended in a unique way from $\mathcal{K}$ to $\mathcal{R}$ such that the extended functionals are additive, i.e. for all $K, K' \in \mathcal{R}$ we have $\Theta_j(K \cup K',\cdot) + \Theta_j(K \cap K',\cdot) = \Theta_j(K,\cdot) + \Theta_j(K',\cdot)$ (where the extensions are denoted by the same symbols). These extensions give rise also to additive extensions of the functionals Φ_j, V_j, and k_j.

Theorem 1 can be used to generalize a result of Arnold [2] from $n \leq 3$ to arbitrary n.

Theorem 2. *Let $K \in \mathcal{K}$ and let $0 < \epsilon < \pi/2$ so that $K_\epsilon \in \mathcal{K}$. Then for $j \in \{0,\dots,n\}$ we have $k_j(K_\epsilon) = \sum_{i=0}^{n} \gamma_{nij}(\epsilon)\, k_i(K)$, where $\gamma_{nij}(\epsilon)$ are numbers defined explicitly in* [4], *pp.* 23 *and* 20.

4 Properties of the support measures

The support measures behave nicely under polarity, as shown by the following result.

Theorem 3. *Let $K \in \mathcal{K}$, $\eta \in \mathcal{B}(S^n \times S^n)$ and $j \in \{0,\dots,n-1\}$. Then $\Theta_j(K,\eta) = \Theta_{n-j-1}(K^*,\eta^{-1})$, where $\eta^{-1} := \{(u,x) \in S^n \times S^n : (x,u) \in \eta\}$.*

It is possible to characterize the linear combinations of the support measures by some of their most elementary properties:

Theorem 4. *Let $\psi : \mathcal{P} \times \mathcal{B}(S^n \times S^n) \to \mathbb{R}$ be a map with the following properties:*
(a) *$\psi(P, \cdot)$ is a finite signed measure, concentrated on* Nor P*, for all $P \in \mathcal{P}$.*
(b) *For all $P \in \mathcal{P}$, $\eta \in \mathcal{B}(S^n \times S^n)$ and $\rho \in SO_{n+1}$ we have $\psi(\rho P, \rho\eta) = \psi(P, \eta)$.*
(c) *For $P, P' \in \mathcal{P}$ and $\eta \in \mathcal{B}(S^n \times S^n)$ with $\eta \cap \operatorname{Nor} P = \eta \cap \operatorname{Nor} P'$ we have $\psi(P, \eta) = \psi(P', \eta)$.*
Then there exist constants $c_0, \dots, c_{n-1} \in \mathbb{R}$ with $\psi(P, \cdot) = \sum_{i=0}^{n-1} c_i \Theta_i(P, \cdot)$ for all $P \in \mathcal{P}$.

In a similar way also the positive linear combinations of the curvature measures can be characterized. Here we have to impose the additional postulate of additivity on the functional under consideration.

Theorem 5. *Let $\psi : \mathcal{P} \times \mathcal{B}(S^n) \to \mathbb{R}$ be a map with the following properties:*
(a) *$\psi(P, \cdot)$ is a finite positive measure, concentrated on P, for all $P \in \mathcal{P}$.*
(b) *For all $P \in \mathcal{P}$, $A \in \mathcal{B}(S^n)$ and $\rho \in SO_{n+1}$ we have $\psi(\rho P, \rho A) = \psi(P, A)$.*
(c) *For $P, P' \in \mathcal{P}$ and open subsets $B \subset S^n$ with $P \cap B = P' \cap B$ we have $\psi(P, A) = \psi(P', A)$ for all $A \in \mathcal{B}(S^n)$ with $A \subset B$.*
(d) *For $P, P' \in \mathcal{P}$ with $P \cup P' \in \mathcal{K}$ we have $\psi(P \cup P', \cdot) + \psi(P \cap P', \cdot) = \psi(P, \cdot) + \psi(P', \cdot)$.*
Then there exist constants $c_0, \dots, c_n \geq 0$ with $\psi(P, \cdot) = \sum_{i=0}^{n} c_i \Phi_i(P, \cdot)$ for all $P \in \mathcal{P}$.

The following version of a Gauss–Bonnet theorem for finite unions of spherically convex bodies is a special case of general results of Allendoerfer & Weil [1]. There are different proofs by several other authors. By using an idea of McMullen [6], we gave a proof which is particularly simple. We denote by $\chi(K)$ the Euler characteristic of the set $K \in \mathcal{R}$, i.e. χ is the unique additive map from $\mathcal{R}$ to $\mathbb{R}$ with the properties $\chi(\emptyset) = 0$ and $\chi(K) = 1$ for all $K \in \mathcal{K}$ that are not subspheres.

Theorem 6. *For $K \in \mathcal{R}$ we have $\chi(K) = 2 \sum_{i=0}^{[n/2]} V_{2i}(K)$.*

Our proof is based on the following Lemma, a version of which was stated in [6], p. 249, without proof. If $P \in \mathcal{P}$ and $F \in \mathcal{F}(P)$ we let $\hat{F} := \{x \in P^* : \langle x, y \rangle = 0 \text{ for all } y \in F\}$. By $\mathbf{1}_A$ we denote the indicator function of a set A.

Lemma. *For all $P \in \mathcal{P}$ that are not subspheres the function*

$$\sum_{F \in \mathcal{F}(P)} (-1)^{\dim F} \mathbf{1}_{F \vee (-\hat{F})}$$

vanishes λ^n-almost everywhere.

5 Kinematic formulae for curvature measures

Our version of the principal kinematic formula for curvature measures can be stated as follows:

Theorem 7. *Let $K, K' \in \mathcal{R}$ and $A, B \in \mathcal{B}(S^n)$. Then for all $j \in \{0, \dots, n\}$ we have*

$$\int_{SO_{n+1}} \Phi_j(K \cap \rho K', A \cap \rho B)\, d\nu(\rho) = \sum_{k=j}^{n} \Phi_k(K, A)\Phi_{n+j-k}(K', B)\,.$$

The main auxiliary results used in our proof were Theorem 5, the weak continuity of the curvature measures, and Theorem 6. There are numerous remarkable consequences of Theorem 7, for which we refer to [4], pp. 58 - 66. We wish to state the following new consequences for the curvature vectors k_j.

Corollary. *For $K, K' \in \mathcal{K}$ and $j \in \{0, \dots, n\}$ we have*

$$\int_{SO_{n+1}} k_j(K \cap \rho K')\, d\nu(\rho) = \sum_{l=j}^{n} k_l(K) V_{n+j-l}(K')\,,$$

$$\int_{SO_{n+1}} k_j(K \vee \rho K')\, d\nu(\rho) = \sum_{l=0}^{j-1} \frac{\alpha_l}{\alpha_j}\, k_l(K) V_{j-l-1}(K') \; + k_j(K)\Big(1 - \sum_{i=0}^{n} V_i(K')\Big)\,,$$

where $\alpha_j = j\beta_j\beta_{n-j-1}/((n-j)\beta_{n-j}\beta_{j-1})$, $1 \le j \le n-1$, and $\alpha_n = 1/\alpha_0 = n\beta_n/(2\beta_{n-1})$.

With the help of Theorem 5 one can also prove an abstract version of Theorem 7, which can be stated as follows.

Theorem 8. *Let $\Lambda : \mathcal{K} \times \mathcal{B}(S^n) \to \mathbb{R}$ be a map with the following properties:*
(a) *$\Lambda(K, \cdot)$ is a finite positive measure, concentrated on K, for all $K \in \mathcal{K}$.*
(b) *$\Lambda(K, \cdot)$ is weakly continuous in $K \in \mathcal{K}$.*
(c) *For $K, K' \in \mathcal{K}$ and open $B \subset S^n$ with $K \cap B = K' \cap B$ we have $\Lambda(K, A) = \Lambda(K', A)$ for all $A \in \mathcal{B}(S^n)$ with $A \subset B$.*
(d) *For $K, K' \in \mathcal{K}$ with $K \cup K' \in \mathcal{K}$ we have $\Lambda(K \cup K', \cdot) + \Lambda(K \cap K', \cdot) = \Lambda(K, \cdot) + \Lambda(K', \cdot)$.*
Then for all $K, K' \in \mathcal{K}$ und $A, B \in \mathcal{B}(S^n)$ we have

$$\int_{SO_{n+1}} \Lambda_j(K \cap \rho K', A \cap \rho B)\, d\nu(\rho) = \sum_{k=j}^{n} \Lambda_k(K, A)\Phi_{n+j-k}(K', B)$$

for all $j \in \{0, \dots, n\}$, where $\Lambda_j(K, \cdot)$ is defined by $\Lambda_j(K, A) := \int_{\mathcal{S}_{n-j}} \Lambda(K \cap S, A)\, d\nu_{n-j}(S)$.

The corresponding result in Euclidean space is due to Schneider [8]. Our method of proof can also be applied in the Euclidean case, where it gives a more direct and simpler proof than the one presented in [8].

6 Kinematic formulae for support measures

There is a generalization of Theorem 7 from curvature measures to support measures. In order to formulate this result, we must define a law of composition between two subsets of

$S^n \times S^n$ which is adapted to the intersection of two convex bodies. For $\eta, \eta' \subset S^n \times S^n$ we let

$$\eta \wedge \eta' := \{(x,u) \in S^n \times S^n \;:\; u \in [u_1,u_2] \text{ with } u_1,u_2 \in S^n,\, u_1 \neq -u_2, \\ (x,u_1) \in \eta,\, (x,u_2) \in \eta'\}\,,$$

where $[u_1, u_2]$ is the closed spherical segment joining the points $u_1, u_2 \in S^n$, $u_1 \neq -u_2$. In the following we assume the measures $\Theta_j(K,\cdot)$ to be complete (we do not introduce new symbols for the completions). The spherical principal kinematic formula for support measures of convex bodies can now be stated as follows.

Theorem 9. *Let $K, K' \in \mathcal{K}$ and $\eta \in \mathcal{B}(\mathrm{Nor}\, K)$, $\eta' \in \mathcal{B}(\mathrm{Nor}\, K')$. Then for $j \in \{0,\dots,n-1\}$ we have*

$$\int_{SO_{n+1}} \Theta_j(K \cap \rho K', \eta \wedge \rho\eta')\, d\nu(\rho) = \sum_{k=j+1}^{n-1} \Theta_k(K,\eta)\Theta_{n+j-k}(K',\eta')\,.$$

If we define the set Nor K also for $K \in \mathcal{R}$ in an appropriate way, Theorem 9 remains valid also for sets $K, K' \in \mathcal{R}$. Since according to Theorem 3 the support measures behave well under polarity, we can infer the following dualized version from Theorem 9. Here we denote for $\eta, \eta' \subset S^n \times S^n$

$$\eta \vee \eta' := (\eta^{-1} \wedge \eta'^{-1})^{-1} = \{(x,u) \in S^n \times S^n \;:\; x \in [x_1,x_2] \text{ with } x_1,x_2 \in S^n, \\ x_1 \neq -x_2,\, (x_1,u) \in \eta,\, (x_2,u) \in \eta'\}\,.$$

Theorem 10. *Let $K, K' \in \mathcal{K}$ and $\eta \in \mathcal{B}(\mathrm{Nor}\, K), \eta' \in \mathcal{B}(\mathrm{Nor}\, K')$. Then for $j \in \{0,\dots,n-1\}$ we have*

$$\int_{SO_{n+1}} \Theta_j(K \vee \rho K', \eta \vee \rho\eta')\, d\nu(\rho) = \sum_{k=0}^{j-1} \Theta_k(K,\eta)\Theta_{j-k-1}(K',\eta')\,.$$

The proof of Theorem 9 is considerably more difficult than that of Theorem 7. We want to state one major lemma, which can be derived from a result of Schneider ([7], Corollary 2.3.11). For $K \in \mathcal{K}$ we let $N(K,x) := \{y \in K^* : \langle x,y\rangle = 0\}$, and we denote the linear hull operation by lin.

Lemma. *Let $K, K' \in \mathcal{K}$. Then for ν-almost all $\rho \in SO_{n+1}$ we have*

$$\mathrm{lin}\; N(K,x) \cap \mathrm{lin}\; N(\rho K',x) = \{0\}$$

for all $x \in K \cap \rho K'$.

The Euclidean counterpart to Theorem 9 could be proved only in the case that one of the convex bodies K, K' is a polytope, since in the general case the Euclidean version of the Lemma turned out to be difficult. Possibly the methods of Ewald et al. [3] and Zalgaller [11] might be applied to solve this problem. We proved the analog of Theorem 9 in the case where the moving object is an affine subspace; this a very general local version of the Crofton formula. A special case of this formula gives rise to a new intuitive interpretation of the area measures of convex bodies, introduced by Aleksandrov, Fenchel, and Jessen. Euclidean analogs to Theorem 10 are not known up to now.

From Theorem 9 a projection formula for the support measures can be derived. If $K \in \mathcal{K}$ is a convex body and $S \in \mathcal{S}_j$, $0 \leq j \leq n-1$, is a subsphere, $K|S := (K \vee S^*) \cap S \in \mathcal{K}$ is the projection of K in S. If $x \notin S^*$ we let $x|S := p(S,x)$ be the projection of x in S, in case $x \in S^*$ the expression $x|S$ shall be undefined. For $\eta \subset S^n \times S^n$ we let

$$\eta|S := \{(x|S, u) \in S^n \times S^n : (x,u) \in \eta, x \notin S^*, u \in S\}\,.$$

In the following theorem, $\Theta_j^{(S)}(K,\cdot)$ denotes the support measure of a convex body $K \subset S$, computed in the subsphere S.

Theorem 11. *Let $K \in \mathcal{K}$, $q \in \{1,\ldots,n-1\}$, $j \in \{0,\ldots,q-1\}$ and $\eta \in \mathcal{B}(\mathrm{Nor}\, K)$. Then*

$$\int_{\mathcal{S}_q} \Theta_j^{(S)}(K|S, \eta|S)\, d\nu_q(S) = \Theta_j(K,\eta)\,.$$

The Euclidean counterpart to Theorem 11 is due to Schneider (see [7], Theorem 4.5.10).

7 Distance integrals

Let $0 < \epsilon < \pi/2$ and let $K, K' \in \mathcal{K}\backslash\{\emptyset\}$ such that $K_\epsilon \in \mathcal{K}$ or $K'_\epsilon \in \mathcal{K}$. Then for all $\rho \in SO_{n+1}$ such that the minimal distance $r(K,\rho K') := \min\{d(x,y) : x \in K, y \in \rho K'\}$ of K and $\rho K'$ satisfies $0 < r(K,\rho K') < \epsilon$ there are unique points $x(K,\rho K') \in K$, $x(\rho K', K) \in \rho K'$ realizing the minimal distance: $d(x(K,\rho K'), x(\rho K', K)) = r(K,\rho K')$. Adding the definitions $u(K,\rho K') := u(K, x(\rho K', K))$, $u(\rho K', K) := u(\rho K', x(K,\rho K'))$, we can ask for the measure of the set

$$\begin{aligned} L_\epsilon(K,K',\eta,\eta') := \{\rho \in SO_{n+1} : 0 < r(K,\rho K') < \epsilon,\\ (x(K,\rho K'), u(K,\rho K')) \in \eta,\ (x(\rho K', K), u(\rho K', K)) \in \rho\eta'\}\,, \end{aligned}$$

where η and η' are Borel subsets of $S^n \times S^n$. This measure is computed in our last result in terms of the support measures of K and K':

Theorem 12. *Let $0 < \epsilon < \pi/2$, and let $K, K' \in \mathcal{K}\backslash\{\emptyset\}$ such that $K_\epsilon \in \mathcal{K}$ or $K'_\epsilon \in \mathcal{K}$. Then for $\eta, \eta' \in \mathcal{B}(S^n \times S^n)$ we have*

$\nu(L_\epsilon(K,K',\eta,\eta'))$

$$= \sum_{k=0}^{n-1} \sum_{l=0}^{k} \sum_{m=0}^{n-k-1} \alpha_{nklm} \Theta_{n-k+l-m-1}(K, \eta) \Theta_{n-l-m-1}(K', \eta') \int_0^{\epsilon} \cos^k t \sin^{n-k-1} t \, dt \,,$$

where

$$\alpha_{nklm} := (-1)^{n-k-m-1} \frac{\binom{n-1}{k}\binom{k}{l}\binom{n-k-1}{m}}{\binom{n-1}{k-l+m}\binom{n-1}{l+m}} \frac{\beta_{k-l+m}\beta_{n-k+l-m-1}\beta_{l+m}\beta_{n-l-m-1}}{\beta_n \beta_{n-1}} \,.$$

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