A generalization of intersection formulae of integral geometry

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Abstract. We establish extensions of the Crofton formula and, under some restrictions, of the principal kinematic formula of integral geometry from curvature measures to generalized curvature measures of convex bodies. We also treat versions for finite unions of convex bodies. As a consequence, we get a new intuitive interpretation of the area measures of Aleksandrov and Fenchel–Jessen.

Key words: Convex bodies, generalized curvature measures, boundary structure, Crofton formula, principal kinematic formula.

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The subject of this paper is the generalization of two integral geometric intersection formulae for curvature measures of convex bodies: the Crofton formula and the principal kinematic formula. The curvature measures of Federer are replaced by the so-called generalized curvature measures, which are concentrated on the set of all support elements of a convex body, and which for this reason we will also call support measures. The proofs of these extensions depend on certain easy-to-state assertions about the boundary structure of convex bodies. For one of these assertions, we were only able to give a proof under some restrictions on the convex bodies under consideration. This results in corresponding limitations for our generalized principal kinematic formula. We strongly conjecture that in fact these restrictions are not necessary. Our version of the Crofton formula, which can be proved without any restrictions, gives rise to a new intuitive interpretation of the support measures and especially of the area measures of convex bodies. We also treat extensions to the convex ring, the set of all finite unions of convex bodies. For analogous results in spherical space, see my thesis [1], which also contains the results of the present article. The paper [2] is a summary of [1]. It remains open whether there exist extensions to classes of more general sets, as considered in the case of the curvature measures, e.g., by Rother & Zähle [3]. For a recent survey on integral geometry of convex bodies, see Schneider & Wieacker [7].

1 Preliminaries

We work in n-dimensional Euclidean space $\mathbb{R}^n$ with standard inner product $\langle\cdot,\cdot\rangle$, induced norm $\|\cdot\|$ and origin o. The linear and affine hull operation is denoted by lin and aff, respectively. $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ is the unit sphere. Let $\mathcal{K}$ be the space of all convex bodies, i.e. the set of all compact, convex subsets of $\mathbb{R}^n$ (including the empty set), topologized as usual. The interior of a convex body K is written int K, the relative interior relint K, the boundary bd K, the dimension $\dim K$. Let $\mathcal{P} \subset \mathcal{K}$ be the set of all polytopes, i.e. the convex hulls of finite sets.

For $K \in \mathcal{K}\backslash\{\emptyset\}$ and $x \in \mathbb{R}^n$ let $d(K,x)$ be the Euclidean distance of x to K, and let $p(K,x) \in K$ be the metric projection of x in K, i.e. the point in K nearest to x. If $d(K,x) > 0$, let $u(K,x) := (x - p(K,x))/d(K,x)$ be the outer unit normal vector of K pointing to x. We let $\Sigma := \mathbb{R}^n \times S^{n-1}$ and $\operatorname{Nor} K := \{(p(K,x), u(K,x)) \in \Sigma : x \in \mathbb{R}^n\backslash K\}$ for $K \in \mathcal{K}\backslash\{\emptyset\}$ and in addition $\operatorname{Nor} \emptyset := \emptyset$. We refer to the elements of $\operatorname{Nor} K$ as the support elements of K. The subset $\operatorname{Nor} K \subset \Sigma$ is compact. For $K \in \mathcal{K}$ and a face F of K let $N(K,F)$ be the normal cone of K at F, i.e. the convex cone

$$N(K,x) := \{u \in \mathbb{R}^n : \langle u, x\rangle \geq \langle u, y\rangle \text{ for all } y \in K\},$$

where x is an arbitrary point from the relative interior of F.

For $\epsilon > 0$, $K \in \mathcal{K}\backslash\{\emptyset\}$ and $\eta \in \mathcal{B}(\Sigma)$ (where $\mathcal{B}(X)$ denotes the σ-algebra of all Borel subsets of a topological space X) let

$$M_\epsilon(K,\eta) := \{x \in \mathbb{R}^n : 0 < d(K,x) \leq \epsilon, (p(K,x), u(K,x)) \in \eta\}$$

be the local parallel set and let $\mu_\epsilon(K,\eta) := \lambda^n(M_\epsilon(K,\eta))$, where λ^n is Lebesgue measure on $\mathcal{B}(\mathbb{R}^n)$. According to the Steiner formula $\mu_\epsilon(K,\eta)$ is a polynomial in ϵ,

$$\mu_\epsilon(K,\eta) = \frac{1}{n}\sum_{j=0}^{n-1} \epsilon^{n-j}\binom{n}{j}\Theta_j(K,\eta), \tag{1}$$

cf. Schneider [5], Theorem 4.2.1. The measures $\Theta_j(K,\cdot)$ on $\mathcal{B}(\Sigma)$ defined by the coefficients are the generalized curvature measures of K, which we will briefly call support measures of K. In addition we define $\Theta_j(\emptyset,\cdot) := 0$ and $\mu_\epsilon(\emptyset,\cdot) := 0$. We will mainly work with the renormalized measures $\Lambda_j(K,\cdot)$, defined by

$$n\kappa_{n-j}\Lambda_j(K,\cdot) = \binom{n}{j}\Theta_j(K,\cdot),$$

where κ_{n-j} is the volume of the $(n-j)$-dimensional unit ball. We also refer to the measures $\Lambda_j(K,\cdot)$ as the support measures of K.

An easy computation of $\mu_\epsilon(P,\eta)$ for $P \in \mathcal{P}$ gives

$$\Lambda_j(P,\eta) = \frac{1}{(n-j)\kappa_{n-j}}\sum_{F\in\mathcal{F}_j(P)}\int_F\int_{N(P,F)\cap S^{n-1}} \mathbf{1}_\eta(x,u)\, d\lambda^j(x) d\lambda^{n-j-1}(u), \tag{2}$$

where $\mathcal{F}_j(P)$ is the set of all j-faces of P, $\mathbf{1}_\eta$ is the indicator function of the set $\eta \in \mathcal{B}(\Sigma)$ and λ^j is j-dimensional Hausdorff measure.

If we choose $\epsilon = 1, \ldots, n$ in the Steiner formula, we can solve the resulting system of linear equations and get the representations

$$\Lambda_j(K,\cdot) = \sum_{k=1}^{n} b_{jk}\mu_k(K,\cdot) \tag{3}$$

for all $K \in \mathcal{K}$ with certain real constants b_{jk}.

By the specialization $\Phi_j(K,A) := \Lambda_j(K, A \times S^{n-1})$, $A \in \mathcal{B}(\mathbb{R}^n)$, we get the curvature measures $\Phi_j(K,\cdot)$ of Federer, and $S_j(K,\omega) := \Theta_j(K, \mathbb{R}^n \times \omega)$, $\omega \in \mathcal{B}(S^{n-1})$, defines the area measures of Aleksandrov and Fenchel–Jessen. The global measures $V_j(K) := \Lambda_j(K,\Sigma)$ are the intrinsic volumes of K. For the most important properties of the functionals Θ_j, Φ_j, S_j, and V_j we refer to Schneider [5], Section 4.2. We would like to point out one additional property of the measures $\Lambda_j(K,\cdot)$: The functional Λ_j is locally determined, i.e. for $K, K' \in \mathcal{K}$ and $\eta \in \mathcal{B}(\Sigma)$ with $\eta \cap \operatorname{Nor} K = \eta \cap \operatorname{Nor} K'$ we have $\Lambda_j(K,\eta) = \Lambda_j(K',\eta)$. This follows from (3) and the fact that also the measures $\mu_\epsilon(K,\cdot)$ are locally determined.

The convex ring $\mathcal{R}$ is the set of all finite unions of convex bodies. There exist uniquely determined additive extensions of the support measures Λ_j to $\mathcal{R}$, i.e. the extended functionals (again denoted by Λ_j) have the valuation property

$$\Lambda_j(K \cap K', \cdot) + \Lambda_j(K \cup K', \cdot) = \Lambda_j(K, \cdot) + \Lambda_j(K', \cdot)$$

for all $K, K' \in \mathcal{R}$. For a construction of these extensions see, e.g., Schneider [5], Section 4.4. If $K \in \mathcal{R}$ is represented as a union of the convex bodies $K_1, \ldots, K_m$, then the inclusion-exclusion principle tells us that

$$\Lambda_j(K, \cdot) = \sum_{v \in S(m)} (-1)^{|v|-1} \Lambda_j(K_v, \cdot)\,. \tag{4}$$

Here $S(m)$ is the set of all non-empty subsets of $\{1, \ldots, m\}$, and for $v \in S(m)$ we denote by $|v|$ the cardinality of v and define $K_v := \cap_{i \in v} K_i$.

Let G_n and SO_n be the groups of all proper rigid motions and proper rotations of $\mathbb{R}^n$, respectively, topologized as usual. Let μ and ν be the Haar measures on G_n and SO_n, normalized by $\mu(\{g \in G_n : g(o) \in B^n\}) = \kappa_n$ and $\nu(SO_n) = 1$, respectively. We let $\mathcal{E}_q^n$ denote the set of all affine q-dimensional subspaces of $\mathbb{R}^n$, $q \in \{0, \ldots, n\}$, again equipped with the usual topology. We denote the Haar measure of the homogeneous G_n-space $\mathcal{E}_q^n$ by μ_q, normalized by $\mu_q(\{E \in \mathcal{E}_q^n : E \cap B^n \neq \emptyset\}) = \kappa_{n-q}$.

We now cite the principal kinematic formula and the Crofton formula in the versions for curvature measures of convex bodies. We use the constants α_{njk} defined by

$$\alpha_{njk} := \frac{\binom{k}{j} \kappa_k \kappa_{n+j-k}}{\binom{n}{k-j} \kappa_j \kappa_n} = \frac{\Gamma\left(\frac{k+1}{2}\right) \Gamma\left(\frac{n+j-k+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}\,. \tag{5}$$

Elementary proofs of the following two theorems, which are special cases of results of Federer, can be found, e.g., in [5], Section 4.5.

Theorem 1.1. *Let $K, K' \in \mathcal{K}$, $A \in \mathcal{B}(\mathrm{bd}\ K)$ and $B \in \mathcal{B}(\mathrm{bd}\ K')$. Then for $j \in \{0, \ldots, n-2\}$, we have*

$$\int_{G_n} \Phi_j(K \cap gK', A \cap gB)\, d\mu(g) = \sum_{k=j+1}^{n-1} \alpha_{njk} \Phi_k(K, A) \Phi_{n+j-k}(K', B)\,.$$

Theorem 1.2. *Let $K \in \mathcal{K}$ and $A \in \mathcal{B}(\mathrm{bd}\ K)$. Then for $q \in \{1, \ldots, n-1\}$ and $j \in \{0, \ldots, q-1\}$, we have*

$$\int_{\mathcal{E}_q^n} \Phi_j(K \cap E, A \cap E)\, d\mu_q(E) = \alpha_{njq} \Phi_{n+j-q}(K, A)\,.$$

We remark that the slightly more general variants treated in [5], where arbitrary $A, B \in \mathcal{B}(\mathbb{R}^n)$ are admitted, can easily be deduced from the above assertions with the help of the Fubini theorem.

In our proofs of generalizations of Theorems 1.1 and 1.2 we use the following simple characterization result for the support measures of polytopes. Its proof is short and elementary, and a special feature lies in the fact that the functional under consideration is not postulated to be a valuation. In another context, characterization of the support measures was investigated by Zähle [8].

Lemma 1.3. *Let $\psi : \mathcal{P} \times \mathcal{B}(\Sigma) \to \mathbb{R}$ be a map satisfying the following properties:*

(a) *$\psi(P, \cdot)$ is a (finite) signed measure for all $P \in \mathcal{P}$.*

(b) *We have $\psi(gP, g\eta) = \psi(P, \eta)$ for all $P \in \mathcal{P}$, $\eta \in \mathcal{B}(\Sigma)$ and $g \in G_n$, where $g\eta := \{(gx, g_0 u) \in \Sigma : (x, u) \in \eta\}$ (here $g_0 \in SO_n$ is the rotational part of g).*

(c) *For $\eta \in \mathcal{B}(\Sigma)$ and $P, P' \in \mathcal{P}$ with $\eta \cap \operatorname{Nor} P = \eta \cap \operatorname{Nor} P'$ we have $\psi(P, \eta) = \psi(P', \eta)$.*

Then there are real numbers $c_0, \ldots, c_{n-1}$ such that for all $P \in \mathcal{P}$

$$\psi(P, \cdot) = \sum_{j=0}^{n-1} c_j \Lambda_j(P, \cdot)\,.$$

Proof: To start with we show $\psi(\emptyset, \cdot) \equiv 0$. We define a finite signed measure ρ by $\rho(A) := \psi(\emptyset, A \times S^{n-1})$, $A \in \mathcal{B}(\mathbb{R}^n)$. From (b) we know that ρ is translation invariant, therefore it must be a multiple of Lebesgue measure. But since ρ is finite, the factor involved must be 0. Thus $\psi(\emptyset, \cdot) \equiv 0$.

We now deduce from (c) that $\psi(P, \cdot)$ is concentrated on $\operatorname{Nor} P$ for all $P \in \mathcal{P}$.

Now let $P \in \mathcal{P} \backslash \{\emptyset\}$, $j \in \{0, \ldots, \dim P\}$ and $F \in \mathcal{F}_j(P)$. An easy argument using (a), (b), (c) and the fact that Lebesgue measure and spherical Lebesgue measure can be characterized by their invariance properties yields

$$\psi(P, A \times B) = b_j \lambda^j(A) \lambda^{n-j-1}(B)$$

for every $A \in \mathcal{B}(\operatorname{relint} F)$, $B \in \mathcal{B}(N(P, F) \cap S^{n-1})$ with a real constant b_j depending only on j. Since we have shown that $\psi(P, \cdot)$ is concentrated on $\operatorname{Nor} P$ and since $\operatorname{Nor} P$ is the disjoint union of the sets $(\operatorname{relint} F) \times N(P, F)$, $F \in \mathcal{F}(P)$, we get from equation (2)

$$\begin{aligned} \Psi(P, A \times B) &= \sum_{j=0}^{n-1} b_j \sum_{F \in \mathcal{F}_j(P)} \lambda^j(F \cap A) \lambda^{n-j-1}(N(P, F) \cap B) \\ &= \sum_{j=0}^{n-1} c_j \Lambda_j(P, A \times B) \end{aligned}$$

for all $A \in \mathcal{B}(\mathbb{R}^n)$, $B \in \mathcal{B}(S^{n-1})$, where c_j is defined by $(n-j)\kappa_{n-j} b_j$. In the usual way we derive

$$\psi(P, \eta) = \sum_{j=0}^{n-1} c_j \Lambda_j(P, \eta)$$

for every $\eta \in \mathcal{B}(\Sigma)$: The set of all $\eta \in \mathcal{B}(\Sigma)$ satisfying this equation is a Dynkin system, which contains all sets of the form $A \times B$ with $A \in \mathcal{B}(\mathbb{R}^n)$, $B \in \mathcal{B}(S^{n-1})$; thus it must coincide with $\mathcal{B}(\Sigma)$. ■

2 A problem concerning the boundary structure of convex bodies

Our proofs of the principal kinematic formula and the Crofton formula for support measures use a certain assertion about the boundary structure of convex bodies in an essential way. Unfortunately we could prove this easy-to-state assertion only in some special cases, so that we can merely formulate it as a conjecture.

Conjecture *Let $K, K' \in \mathcal{K}$ be convex bodies. Then for μ-a.e. $g \in G_n$, we have*

$$\text{lin } N(K,x) \cap \text{lin } N(gK',x) = \{o\}$$

for all $x \in \text{bd } K \cap \text{bd } gK'$.

In favour of this conjecture we can state that it is true if one of the two bodies is a polytope, that it is correct in dimensions two and three and that the corresponding assertion in spherical space is true in all dimensions (for this result we refer to [1], p. 78).

It is not difficult to see that the assertion is true for $n = 2$. In the case $n = 3$ we can argue as follows. Let $(x, L) \in \mathbb{R}^n \times \mathcal{L}_q^n$, $(x', L') \in \mathbb{R}^n \times \mathcal{L}_r^n$ with $q, r \in \{1, \ldots, n-1\}$. The set of all motions $g \in G_n$ with $x = gx'$ and $\dim(L \cap g_0 L') \geq 1$, where $g_0 \in SO_n$ is the rotational part of g, is a compact m-dimensional submanifold of G_n, where $m = \min\{n(n-1)/2, n(n-1)/2 + q + r - n - 1\}$. In the case $n = 3$ it is not difficult to show that the set $\{(x, L) \in \mathbb{R}^n \times \mathcal{L}_q^n : \text{lin } N(K,x) = L\}$ can be written as a union of sets A_i, $i \in \mathbb{N}$, such that each A_i can be covered, for all sufficiently small $\epsilon > 0$, by at most $c_i \epsilon^{-(n-q)}$ subsets of $\mathbb{R}^n \times \mathcal{L}_q^n$ of diameter ϵ (here the numbers c_i are independent of ϵ, and the term "diameter" refers to the product metric on $\mathbb{R}^n \times \mathcal{L}_q^n$, where the metric on $\mathcal{L}_q^n$ is induced by a rotation invariant Riemannian metric). The methods used in Schneider [5], pp. 90 - 93, then show that the set

$$\begin{aligned} \{g \in G_n \quad : \quad & \text{there is an } x \in \text{bd } K \cap \text{bd } gK' \text{ with } L := \text{lin } N(K,x) \in \mathcal{L}_q^n, \\ & L' := \text{lin } N(gK',x) \in \mathcal{L}_r^n \text{ and } \dim(L \cap L') \geq 1\} \end{aligned}$$

has σ-finite Hausdorff measure of dimension $(n-q)+(n-r)+(n(n-1)/2+q+r-n-1) = n(n+1)/2-1$. Since G_n has Hausdorff dimension $n(n+1)/2$, our conjecture is true for $n = 3$. (Before this argument was found, a different proof for the case $n = 3$ was communicated to me by Rolf Schneider, see the proof of Satz 8.3.2 in [1].)

This argument can be extended to general dimensions if the following question has a positive answer. For $q \in \{1, \ldots, n-2\}$, let $x_0, \ldots, x_q \in \mathbb{R}^n$ be affinely independent, let L be the $(n-q)$-dimensional linear subspace orthogonal to aff $\{x_0, \ldots, x_q\}$, let $E_i := L + x_i \in \mathcal{E}_{n-q}^n$, let $\emptyset \neq K_i \subset E_i$ be convex bodies (which may be $(n-q)$-dimensional parallel bodies of some given convex bodies in the flats E_i), and let $K := \text{conv } (K_0 \cup \cdots \cup K_q)$. Define

$$\mathcal{A} := \left\{ \text{aff } (H \cap K) \in \mathcal{E}_q^n : H \text{ is a support hyperplane of } K, \ \text{card}(H \cap K_i) = 1 \ \forall i \right\}.$$

Equip the space $\mathcal{E}_q^n$ with a metric which is induced by a motion invariant Riemannian metric. Then our question is as follows. Can $\mathcal{A}$ be covered, for all sufficiently small $\epsilon > 0$, by at most $c\,\epsilon^{-(n-q-1)}$ subsets of $\mathcal{E}_q^n$ of diameter ϵ, where c is independent of ϵ ?

In this article we give a proof of our conjecture for the case that one of the bodies is a polytope. This proof uses a deeper result of Zalgaller [9].

Theorem 2.1. *The conjecture is true if one of the convex bodies K, K' is a polytope.*

Theorem 2.1 is a consequence of the following result, which we want to state as a theorem as well. Here we denote by $E^\perp$ the linear subspace totally orthogonal to the affine subspace E.

Theorem 2.2. *Let $K \in \mathcal{K}$ and let $q \in \{0, \ldots, n-1\}$. Then for μ_q-a.e. $E \in \mathcal{E}_q^n$, we have*

$$E^\perp \cap \text{lin } N(K,x) = \{o\}$$

for all $x \in E \cap \text{bd } K$.

The proofs of Theorems 2.1 and 2.2 will be given in Section 4.

3 Intersection formulae for support measures

In order to generalize the principal kinematic formula from curvature measures to support measures, we have to define a law of composition between two subsets of Σ which is adapted for the intersection of two convex bodies. The natural definition of such a law of composition seems to be the following: For $\eta, \eta' \subset \Sigma$ we let

$$\eta \wedge \eta' := \big\{(x,u) \in \Sigma \quad : \quad \text{there are } u_1, u_2 \in S^{n-1} \text{ with } (x,u_1) \in \eta,\, (x,u_2) \in \eta',\, u \in \text{pos}\, \{u_1,u_2\}\big\}\,,$$

where pos $\{u_1, u_2\} := \{\lambda_1 u_1 + \lambda_2 u_2 : \lambda_1, \lambda_2 \geq 0\}$ is the positive hull of the set $\{u_1, u_2\}$.

In the following it is necessary to assume that the support measures are complete measures (we do not introduce new symbols for the completions).

In the case of the principal kinematic formula we have to impose restrictions on the convex bodies under consideration, since the general conjecture formulated in the last section remains open. We call a pair K, K' of convex bodies *admissible*, if this conjecture is true for these two bodies.

Our principal kinematic formula for support measures can now be stated as follows. Note that the constants α_{njk} have been defined by (5). For a more satisfactory analogue in spherical space, see [1], Satz 6.1.1, or [2], Theorem 9.

Theorem 3.1. *Let $K, K' \in \mathcal{K}$ be an admissible pair of convex bodies, and let $\eta \in \mathcal{B}(\text{Nor } K), \eta' \in \mathcal{B}(\text{Nor } K')$. Then*

$$\int\limits_{G_n} \Lambda_j(K \cap gK', \eta \wedge g\eta')\, d\mu(g) = \sum_{k=j+1}^{n-1} \alpha_{njk} \Lambda_k(K,\eta) \Lambda_{n+j-k}(K',\eta')$$

for $j \in \{0, \ldots, n-2\}$.

If $K, K' \in \mathcal{K}$ are convex bodies which do not touch each other and A, B are Borel subsets of bd K, bd K', respectively, then it is easy to see that

$$\text{Nor } (K \cap K') \cap \big((A \cap B) \times S^{n-1}\big) = \big(\text{Nor } K \cap (A \times S^{n-1})\big) \wedge \big(\text{Nor } K' \cap (B \times S^{n-1})\big)\,.$$

Since K and gK' do not touch each other for μ-a.e. $g \in G_n$ (see, e.g., [6], Hilfssatz 2.1.4), in the case $\eta = \text{Nor } K \cap (A \times S^{n-1})$, $\eta' = \text{Nor } K' \cap (B \times S^{n-1})$ Theorem 3.1 thus reduces, for admissible $K, K' \in \mathcal{K}$, to the principal kinematic formula for curvature measures.

The following generalization of Theorem 1.2 can be proved without any restrictions imposed on the convex bodies under consideration. The appropriate law of composition between a subset of Σ and an affine subspace is defined as follows: For $\eta \subset \Sigma$ and $E \in \mathcal{E}_q^n$, $q \in \{1, \ldots, n-1\}$, we let

$$\eta \wedge E := \big\{(x,u) \in \Sigma \quad : \quad \text{there are } u_1, u_2 \in S^{n-1} \text{ with } (x,u_1) \in \eta,\, x \in E,\, u_2 \in E^\perp,\, u \in \text{pos}\, \{u_1,u_2\}\big\},$$

where $E^\perp \in \mathcal{L}_{n-q}^n$ is the linear subspace orthogonal to E.

Theorem 3.2. *Let $K \in \mathcal{K}$ and $q \in \{1, \ldots, n-1\}$. Then for every $\eta \in \mathcal{B}(\text{Nor } K)$ we have*

$$\int\limits_{\mathcal{E}_q^n} \Lambda_j(K \cap E, \eta \wedge E)\, d\mu_q(E) = \alpha_{njq} \Lambda_{n+j-q}(K,\eta)$$

for all $j \in \{0, \ldots, q-1\}$.

Remark 1*:* It was indicated in [1], p. 111, how Theorem 3.2 for $q = 2$ and $j = 0, 1$ can be used to show that the formula of Theorem 3.1 is true in the case $j = n - 2$ for arbitrary convex bodies K, K'.

Remark 2*:* We note that in the case $j = 0$ Theorem 3.2 yields a new intuitive interpretation of the support measures, since it is a known fact that

$$\Theta_0(K, \eta) = n\kappa_n \Lambda_0(K, \eta) = \lambda^{n-1}(\{u \in S^{n-1} : (x, u) \in \eta \cap \mathrm{Nor}\, K \text{ for an } x \in \mathrm{bd}\, K\})$$

for all $K \in \mathcal{K}$ and $\eta \in \mathcal{B}(\Sigma)$, see Schneider [4], p. 120, (4.5). In particular we have

$$S_q(K, \omega) = \frac{1}{\kappa_q} \int\limits_{\mathcal{E}^n_{n-q}} \Theta_0(K \cap E, (\mathrm{Nor}\, K \cap (\mathbb{R}^n \times \omega)) \wedge E)\, d\mu_{n-q}(E) \tag{6}$$

for all Borel sets $\omega \subset S^{n-1}$ and all $q \in \{1, \ldots, n - 1\}$, which gives a new integral geometric interpretation of the area measures.

The validity of Theorems 3.1 and 3.2 can be extended from convex bodies to elements of the convex ring. In order to formulate these results it is necessary to define the set Nor K of all support elements also for sets K of the convex ring $\mathcal{R}$. It is possible to give an adequate definition with the help of the index function treated by Schneider [5], Section 4.4, but we prefer to give a simple definition which does not require additional concepts. If $K \in \mathcal{R}$ is an element of the convex ring, let $I(K)$ be the set of all sequences $(K_i)_{i\in\mathbb{N}}$ in $\mathcal{K}$ with $K = \cup_{i=1}^{\infty} K_i$ and $K_i = \emptyset$ for almost all $i \in \mathbb{N}$. Let further $S(\mathbb{N})$ be the set of all non-empty subsets of $\mathbb{N}$. We now define

$$\mathrm{Nor}\, K := \bigcap_{(K_i)\in I(K)} \ \bigcup_{v\in S(\mathbb{N})} \mathrm{Nor}\, (\cap_{i\in v} K_i)\, .$$

For $K \in \mathcal{K}$ this is obviously consistent with the previous definition. The set Nor K is compact for all $K \in \mathcal{R}$.

Since we were not able to prove the conjecture of Section 2 in its full generality, we can show the principal kinematic formula for support measures of sets of the convex ring only in a restricted version. We call a pair $K, K' \in \mathcal{R}$ *admissible*, if there are representations $K = K_1 \cup \cdots \cup K_m$, $K' = K'_1 \cup \cdots \cup K'_{m'}$ with $K_1, \ldots, K_m$, $K'_1, \ldots, K'_{m'} \in \mathcal{K}$ such that for every $v \in S(v)$ and every $v' \in S(m')$ the pair $K_v, K'_{v'}$ of convex bodies is admissible in the sense introduced above.

We can now state our results as follows.

Theorem 3.3. *Theorem* 3.1 *remains true if the pair* K, K' *is replaced by an admissible pair of elements of the convex ring* $\mathcal{R}$*.*

Theorem 3.4. *Theorem* 3.2 *is valid also for sets* $K \in \mathcal{R}$*.*

The results of this section will be proved in Section 5.

4 Proofs of the results stated in Section 2

Proof of Theorem 2.2*:* Let $K \in \mathcal{K}$ and $q \in \{0, \ldots, n-1\}$. We say that a flat $E \in \mathcal{E}^n_q$ is in K-general position, if $E^\perp \cap \mathrm{lin}\, N(K, x) = \{o\}$ for every $x \in E \cap \mathrm{bd}\, K$. Let $\mathcal{A}$ be the set of all $E \in \mathcal{E}^n_q$ which are not in K-general position. $\mathcal{A}$ is contained in $\mathcal{B}(\mathcal{E}^n_q)$ since $\mathcal{A} = \cup_{m=1}^{\infty} A_m$ with

$$\begin{aligned} A_m := \{E \in \mathcal{E}^n_q \quad : \quad & \text{there are } x \in E \cap \mathrm{bd}\, K, u_1, u_2 \in N(K, x) \cap S^{n-1} \text{ and } \lambda_1, \lambda_2 \in \mathbb{R} \\ & \text{with } |\lambda_1|, |\lambda_2| \le m \text{ and } \lambda_1 u_1 + \lambda_2 u_2 \in E^\perp \cap S^{n-1}\} \end{aligned}$$

and each A_m is closed (I thank Professor Rolf Schneider for pointing this out to me).

We denote the set of all linear subspaces of dimension q by $\mathcal{L}_q^n$ and the rotation invariant probability measure on $\mathcal{B}(\mathcal{L}_q^n)$ by ν_q. If $f : \mathcal{E}_q^n \to \mathbb{R}$ is a nonnegative Borel measurable funtion, we have

$$\int\limits_{\mathcal{E}_q^n} f \, d\mu_q = \int\limits_{\mathcal{L}_q^n} \int\limits_{L^\perp} f(L+x) \, d\lambda^{L^\perp}(x) d\nu_q(L) \,,$$

where $\lambda^{L^\perp}$ is $(n-q)$-dimensional Lebesgue measure in the subspace $L^\perp \in \mathcal{L}_{n-q}^n$.

At first we assume $o \in \operatorname{int} K$ and show $\nu_q(\mathcal{A} \cap \mathcal{L}_q^n) = 0$. Let $L \in \mathcal{A} \cap \mathcal{L}_q^n$. Then there exists an $x \in L \cap \operatorname{bd} K$ with $\dim(L^\perp \cap N(K,x)) \geq 1$. Thus there are $v_1, \dots, v_n \in N(K,x)$ and real numbers $\lambda_1, \dots, \lambda_n$ with $\sum_{i=1}^n \lambda_i v_i \in L^\perp \backslash \{o\}$. Let w_1 be the sum of all $\lambda_i v_i$ with $\lambda_i > 0$ and let $-w_2$ be the sum of all $\lambda_i v_i$ with $\lambda_i < 0$. Then w_1, w_2 define a segment in $N(K,x)$ which is parallel to $L^\perp$. Let H be the supporting hyperplane of the polar body

$$K^* := \{y \in \mathbb{R}^n : \langle y, z \rangle \leq 1 \text{ for all } z \in K\}$$

of K at its outer normal vector x and let $F := H \cap K^*$ be the corresponding support set. We have $y \in F$ if and only if $\langle x, y \rangle = \max\{\langle y, z \rangle : z \in K\} = 1$. Therefore $F = H \cap N(K,x)$. Let $\lambda > 0$ with $\lambda w_1 \in F$. Since $L^\perp$ is parallel to H we have $\lambda w_2 \in F$. Thus $\lambda w_1, \lambda w_2$ define a segment in the boundary of K^* which is parallel to $L^\perp$. According to a result of Zalgaller [9] (cf. the formulation in [5], pp. 93 - 94) the set of all $L \in \mathcal{L}_q^n$ for which there is a segment in the boundary of K^* that is parallel to $L^\perp$ has ν_q-measure zero. Hence ν_q-almost all $L \in \mathcal{L}_q^n$ are in K-general position.

Now let K be an arbitrary convex body with non-empty interior. Let $B_0 := \{E \in \mathcal{E}_q^n : E \text{ touches } K\}$ and $B_m := \{E \in \mathcal{E}_q^n : E \in \mathcal{A} \backslash B_0,\, V_q(K \cap E) \geq 1/m\}$ for $m \in \mathbb{N}$. The set B_0 is closed, and the sets B_m, $m \geq 1$, are Borel sets since the map $E \mapsto V_q(K \cap E)$ is continuous on $\mathcal{E}_q^n \backslash B_0$ (this follows from [5], Theorem 1.8.8, and the continuity of the volume functional). We have $\mathcal{A} = \cup_{m=0}^\infty B_m$, and it is easy to see that $\mu_q(B_0) = 0$. For $m \geq 1$ we deduce from the Fubini theorem

$$\begin{aligned}
\mu_q(B_m) &= \int\limits_{\mathcal{L}_q^n} \int\limits_{L^\perp} \mathbf{1}_{B_m}(L+y) \, d\lambda^{L^\perp}(y) d\nu_q(L) \\
&= \int\limits_{\mathcal{L}_q^n} \int\limits_{\{y \in L^\perp : L+y \in B_m\}} V_q(K \cap (L+y))^{-1} \int\limits_{L} \mathbf{1}_K(y+z) \, d\lambda^L(z) d\lambda^{L^\perp}(y) d\nu_q(L) \\
&\leq m \int\limits_{\mathcal{L}_q^n} \int\limits_{L^\perp} \int\limits_{L} \mathbf{1}_{B_m}(L+y+z) \mathbf{1}_K(y+z) \, d\lambda^L(z) d\lambda^{L^\perp}(y) d\nu_q(L) \\
&= m \int\limits_{K} \int\limits_{\mathcal{L}_q^n} \mathbf{1}_{B_m}(L+x) \, d\nu_q(L) d\lambda^n(x) \,.
\end{aligned}$$

It follows from what we have shown above that

$$\int\limits_{\mathcal{L}_q^n} \mathbf{1}_{B_m}(L+x) \, d\nu_q(L) = 0$$

for all $x \in \operatorname{int} K$. Thus we deduce $\mu_q(B_m) = 0$ and therefore $\mu_q(\mathcal{A}) = 0$.

Finally let $r := \dim K < n$. If $q + r \leq n - 1$, then $K \cap E = \emptyset$ for μ_q-almost all $E \in \mathcal{E}_q^n$ and our assertion is trivial. Let $q + r \geq n$ and let $F := \operatorname{aff} K \in \mathcal{E}_r^n$. It follows from [5], Lemma 4.5.1, that $E^\perp \cap F^\perp = \{o\}$ for μ_q-almost all $E \in \mathcal{E}_q^n$. For $E \in \mathcal{E}_q^n$ with $E^\perp \cap F^\perp = \{o\}$ the flat E is in

K-general position if and only if the $(q+r-n)$-dimensional flat $E\cap F$ is in K-general position in the subspace F (this follows from the easily verified equivalence

$$(E\cap F)^{\perp}\cap L=\{o\} \iff E^{\perp}\cap \text{lin}\,(F^{\perp}\cup L)=\{o\}$$

for linear subspaces L with $L+x\subset F$ for $x\in F$). The map $f:\{E\in\mathcal{E}_q^n : E^{\perp}\cap F^{\perp}=\{o\}\}\to\mathcal{E}_{q+r-n}^n$, $E\mapsto E\cap F$, is continuous. The image of the restriction of μ_q under f is invariant with respect to motions which fix F, it is finite on compact sets, and it does not vanish identically. The range of f can be identified with $\mathcal{E}_{q+r-n}^r$. Since the Haar measures on $\mathcal{E}_{q+r-n}^r$ differ only by constant positive factors (for a simple proof, see [6], Satz 1.3.4), this image measure must coincide, up to a constant positive factor, with the measure μ_{q+r-n} on $\mathcal{E}_{q+r-n}^r$. Now it follows from our treatment of n-dimensional convex bodies that $\mu_q(\mathcal{A})=0$. ■

Proof of Theorem 2.1*:* Let $K\in\mathcal{K}$ and $P\in\mathcal{P}$. We have lin $N(K,x)\cap$ lin $N(P,x)=\{o\}$ for all $x\in$ bd $K\cap$bd P if the affine hull of every face of P is in K-general position. Thus it is sufficient to show that for every $E\in\mathcal{E}_q^n$, $q\in\{0,\dots,n-1\}$, for μ-almost all $g\in G_n$ the flat gE is in K-general position. Let again $\mathcal{A}$ be the set of all q-flats which are not in K-general position. We have

$$\begin{aligned}
&\mu(\{g\in G_n : gE\in\mathcal{A}\})\\
&\quad=\int_{SO_n}\int_{\mathbb{R}^n}\mathbf{1}_{\mathcal{A}}(\rho(E+x))\,d\lambda^n(x)d\nu(\rho)\\
&\quad=\int_E\int_{SO_n}\int_{E^{\perp}}\mathbf{1}_{\mathcal{A}}(\rho(E+x))\,d\lambda^{E^{\perp}}(x)d\nu(\rho)d\lambda^E(y)\\
&\quad=\int_E\int_{\mathcal{L}_q^n}\int_{L^{\perp}}\mathbf{1}_{\mathcal{A}}(L+x)\,d\lambda^{L^{\perp}}(x)d\nu_q(L)d\lambda^E(y)\\
&\quad=\int_E\mu_q(\mathcal{A})\,d\lambda^E(y)=0\,,
\end{aligned}$$

since $\mu_q(\mathcal{A})=0$ by Theorem 2.2. ■

5 Proofs of the results stated in Section 3

We are now going to prove Theorem 3.1 with the help of a sequence of lemmas. The proof of Theorem 3.2, which relies heavily on Theorem 2.2, can be given along the same lines (alternatively one could deduce Theorem 3.2 from Theorems 3.1 and 2.1, see [1], pp. 122 - 124). For this reason we omit the proof of Theorem 3.2.

In the following we use the abbreviations $[u_1,u_2]:=$ pos $\{u_1,u_2\}\cap S^{n-1}$ and $]u_1,u_2[\;:=[u_1,u_2]\backslash\{u_1,u_2\}$ for $u_1,u_2\in S^{n-1}$.

If $K,K'\in\mathcal{K}$ is an admissible pair of convex bodies and gK' is a congruent copy of K', we say that K and gK' are *in general relative position*, if lin $N(K,x)\cap$ lin $N(gK',x)=\{o\}$ for all $x\in$ bd $K\cap$bd gK'. In the proofs of the subsequent lemmas we often use the following fact. If $K,K'\in\mathcal{K}$ are in general relative position and if $(x,u)\in$ Nor $K\wedge$ Nor K' but $(x,u)\notin$ Nor $K\cup$ Nor K', then there are uniquely determined $u_1,u_2\in S^{n-1}$ with $(x,u_1)\in$ Nor K, $(x,u_2)\in$ Nor K' and $u\in\;]u_1,u_2[$. Indeed, since lin $N(K,x)\cap$ lin $N(K',x)=\{o\}$ we have $u=v_1+v_2$ with uniquely determined $v_1\in N(K,x)$, $v_2\in N(K',x)$, $v_1,v_2\neq o$, and $u_i=v_i/\|v_i\|$, $i=1,2$.

For the purpose of the proof of Theorem 3.1, we introduce a second law of composition between two subsets of Σ. For $\eta, \eta' \subset \Sigma$ we define

$$\eta \sqcap \eta' := \big\{(x,u) \in \Sigma \quad : \quad \text{there are } u_1, u_2 \in S^{n-1} \text{ with } (x,u_1) \in \eta,\ (x,u_2) \in \eta',\ u \in\,]u_1,u_2[\big\}\,.$$

We work with the measures $\mu_\epsilon(K,\cdot)$, the local parallel volumes as defined in Section 1. If not stated otherwise we always assume that K, K' is an admissible pair of convex bodies. Let $\epsilon > 0$.

Lemma 5.1. *Let $\eta \in \mathcal{B}(\mathrm{Nor}\ K)$, $\eta' \in \mathcal{B}(\mathrm{Nor}\ K')$. Then for μ-a.e. $g \in G_n$ we have $\eta \sqcap g\eta' \in \mathcal{B}(\Sigma)$. For $g \in G_n$ there is a set $B_g \in \mathcal{B}(\Sigma)$ with $(\eta \wedge g\eta') \backslash (\eta \sqcap g\eta') \subset B_g$ such that $\mu_\epsilon(K \cap gK', B_g) = 0$ for μ-a.e. $g \in G_n$.*

Proof: Let $g \in G_n$ such that K and gK' are in general relative position. If $(x,u) \in X := (\mathrm{Nor}\ K \wedge \mathrm{Nor}\ gK') \backslash (\mathrm{Nor}\ K \cup \mathrm{Nor}\ gK')$, then, as mentioned above, there exist uniquely determined $u_1, u_2 \in S^{n-1}$ with $(x,u_1) \in \mathrm{Nor}\ K$, $(x,u_2) \in \mathrm{Nor}\ gK'$ and $u \in\,]u_1,u_2[$. For $i \in \{1,2\}$ we let

$$\pi_i : X \to \Sigma, \quad (x,u) \mapsto (x,u_i),$$

so $u_1, u_2 \in S^{n-1}$ are determined by $(x,u_1) \in \mathrm{Nor}\ K$, $(x,u_2) \in \mathrm{Nor}\ gK'$ and $u \in\,]u_1,u_2[$. The maps π_1, π_2 are continuous: If $(x_j,u_j)_{j\in\mathbb{N}}$ is a sequence in X with $\lim_{j\to\infty}(x_j,u_j) = (x,u) \in X$, such that $(x_j,v_j) := \pi_1(x_j,u_j)$ does not converge to $(x,v_0) := \pi_1(x,u)$, then there is an increasing sequence $(i_j)_{j\in\mathbb{N}}$ in $\mathbb{N}$ with $(v_{i_j})_{j\in\mathbb{N}}$ converging to a $v \in S^{n-1}$ with $v \neq v_0$. Let w be an accumulation point of the sequence $(w_{i_j})_{j\in\mathbb{N}}$, where $(x_j,w_j) := \pi_2(x_j,u_j)$. Because of $(x,v) \in \mathrm{Nor}\ K$, $(x,w) \in \mathrm{Nor}\ gK'$ we have $u \in\,]v,w[$ and therefore $\pi_1(x,u) = (x,v) \neq (x,v_0) = \pi_1(x,u)$. This contradiction shows the continuity of π_1, and the same argument yields the continuity of π_2.

We have $\eta \sqcap g\eta' = \pi_1^{-1}(\eta) \cap \pi_2^{-1}(g\eta')$: If $(x,u) \in \eta \sqcap g\eta'$, then there exist $u_1, u_2 \in S^{n-1}$ with $(x,u_1) \in \eta, (x,u_2) \in g\eta'$ and $u \in\,]u_1,u_2[$. Since K, gK' are in general relative position, we have $(x,u) \notin \mathrm{Nor}\ K \cup \mathrm{Nor}\ gK'$ and hence $(x,u) \in \pi_1^{-1}(\eta) \cap \pi_2^{-1}(g\eta')$. For the reverse inclusion let $(x,u) \in \pi_1^{-1}(\eta) \cap \pi_2^{-1}(g\eta')$. Then on the one hand there exist $v_1, v_2 \in S^{n-1}$ with $(x,v_1) \in \eta$, $(x,v_2) \in \mathrm{Nor}\ gK'$ and $u \in\,]v_1,v_2[$, on the other hand $w_1, w_2 \in S^{n-1}$ with $(x,w_1) \in \mathrm{Nor}\ K$, $(x,w_2) \in g\eta'$ and $u \in\,]w_1,w_2[$. Because of $\mathrm{lin}\ N(K,x) \cap \mathrm{lin}\ N(gK',x) = \{o\}$ it follows $v_1 = w_1$, $v_2 = w_2$ and therefore $(x,u) \in \eta \sqcap g\eta'$.

Because of $X \in \mathcal{B}(\Sigma)$ it now follows that $\eta \sqcap g\eta' \in \mathcal{B}(X) \subset \mathcal{B}(\Sigma)$.

Now let $g \in G_n$ be arbitrary. The set

$$B_g := (\mathrm{Nor}\ K \cup \mathrm{Nor}\ gK') \cap ((\mathrm{bd}\ K \cap \mathrm{bd}\ gK') \times S^{n-1}) \in \mathcal{B}(\Sigma)$$

obviously satifies

$$(\eta \wedge g\eta') \backslash (\eta \sqcap g\eta') \subset B_g\,.$$

Furthermore we have for the local parallel sets

$$M_\epsilon(K \cap gK', B_g) = M_\epsilon(K, \mathrm{bd}\ gK' \times S^{n-1}) \ \cup\ M_\epsilon(gK', \mathrm{bd}\ K \times S^{n-1})$$

and therefore

$$\int_{G_n} \mu_\epsilon(K \cap gK', B_g)\, d\mu(g) \leq \int_{G_n} \Big(\mu_\epsilon(K, \mathrm{bd}\ gK' \times S^{n-1}) + \mu_\epsilon(gK', \mathrm{bd}\ K \times S^{n-1})\Big)\, d\mu(g)\,.$$

An application of the Fubini theorem gives (see, e.g., [6], Satz 1.2.7)

$$\begin{aligned}
&\int\limits_{G_n} \mu_\epsilon(K, \text{bd } gK' \times S^{n-1})\, d\mu(g) \\
&= \int\limits_{SO_n} \int\limits_{\mathbb{R}^n} \mu_\epsilon(K, (\text{bd } \vartheta K' + t) \times S^{n-1})\, d\lambda^n(t) d\nu(\vartheta) \\
&= \mu_\epsilon(K, \Sigma) \int\limits_{SO_n} \lambda^n(\text{bd } \vartheta K')\, d\nu(\vartheta) = 0
\end{aligned}$$

and similarly $\int_{G_n} \mu_\epsilon(gK', \text{bd } K \times S^{n-1}) d\mu(g) = 0$. It follows

$$\int\limits_{G_n} \mu_\epsilon(K \cap gK', B_g)\, d\mu(g) = 0$$

and therefore $\mu_\epsilon(K \cap gK', B_g) = 0$ for μ-a.e. $g \in G_n$, as asserted. ∎

Now it is clear that under the above assumptions the set $\eta \wedge g\eta'$ is $\mu_\epsilon(K \cap gK', \cdot)$-measurable for μ-almost all $g \in G_n$. In the following we assume that the measures $\mu_\epsilon(K \cap gK', \cdot)$ are complete. As a consequence of Lemma 5.1 we have

$$\mu_\epsilon(K \cap gK', \eta \wedge g\eta') = \mu_\epsilon(K \cap gK', \eta \sqcap g\eta')$$

for μ-almost all $g \in G_n$.

Lemma 5.2. *Let $\eta' \in \mathcal{B}(\text{Nor } K')$. Then for μ-a.e. $g \in G_n$, the map*

$$\mathcal{B}(\text{Nor } K) \to \mathbb{R}\,, \qquad \eta \mapsto \mu_\epsilon(K \cap gK', \eta \wedge g\eta')\,,$$

is a finite measure.

Proof: Let $g \in G_n$ such that K and gK' are in general relative position. Let $x \in \text{bd } K \cap \text{bd } gK'$. Because of $\text{lin } N(K, x) \cap \text{lin } N(gK', x) = \{o\}$ we have for all $A, B \subset N(K, x) \cap S^{n-1}$ with $A \cap B = \emptyset$ and all $C \subset N(gK', x) \cap S^{n-1}$

$$\Big((\{x\} \times A) \sqcap (\{x\} \times C)\Big) \cap \Big((\{x\} \times B) \sqcap (\{x\} \times C)\Big) = \emptyset\,.$$

From this it follows generally

$$(\eta_1 \sqcap g\eta') \cap (\eta_2 \sqcap g\eta') = \emptyset$$

for all $\eta_1, \eta_2 \subset \text{Nor } K, \eta_1 \cap \eta_2 = \emptyset$, and $\eta' \subset \text{Nor } K'$. Since $\mu_\epsilon(K \cap gK', \cdot)$ is a finite measure, the assertion follows from Lemma 5.1. ∎

Lemma 5.3. *If $(K_i)_{i\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ with $\lim_{i\to\infty} K_i = K$ such that the pair K_i, K' is admissible for all $i \in \mathbb{N}$, then for μ-a.e. $g \in G_n$*

$$\begin{aligned}
&\liminf_{i\to\infty} \mu_\epsilon\big(K_i \cap gK', (\eta \cap \text{Nor } K_i) \wedge g(\eta' \cap \text{Nor } K')\big) \\
&\qquad \geq \mu_\epsilon\big(K \cap gK', (\eta \cap \text{Nor } K) \wedge g(\eta' \cap \text{Nor } K')\big)
\end{aligned}$$

for all open sets $\eta, \eta' \subset \Sigma$ and

$$\lim_{i\to\infty} \mu_\epsilon\big(K_i \cap gK', \text{Nor } K_i \wedge \text{Nor } gK'\big) = \mu_\epsilon\big(K \cap gK', \text{Nor } K \wedge \text{Nor } gK'\big)\,.$$

Proof: Let η and η' be open subsets of Σ. For μ-a.e. $g \in G_n$ both K, gK' and K_i, gK' for all $i \in \mathbb{N}$ are in general relative position. In the following g is an arbitrary motion having this property. The assertion is trivial if $K \cap gK' = \emptyset$. So let us assume $K \cap gK' \neq \emptyset$. The bodies K, gK' do not touch each other (otherwise they were not in general relative position), therefore we have $\lim_{i\to\infty}(K_i \cap gK') = K \cap gK'$ according to Theorem 1.8.8 in [5]. We show that for all $\delta \in (0, \epsilon)$ the inclusion

$$\begin{aligned} &M_\delta\big(K \cap gK', (\eta \cap \text{Nor } K) \sqcap g(\eta' \cap \text{Nor } K')\big) \\ &\qquad \subset \liminf_{i\to\infty} M_\epsilon\big(K_i \cap gK', (\eta \cap \text{Nor } K_i) \sqcap g(\eta' \cap \text{Nor } K')\big) \end{aligned}$$

holds (as shown in the proof of Lemma 5.1 both $(\eta \cap \text{Nor } K) \sqcap g(\eta' \cap \text{Nor } K')$ and $(\eta \cap \text{Nor } K_i) \sqcap g(\eta' \cap \text{Nor } K')$ are Borel sets).

Assume $x \in M_\epsilon(K \cap gK', (\eta \cap \text{Nor } K) \sqcap g(\eta' \cap \text{Nor } K'))$ with $d(K \cap gK', x) < \epsilon$. For almost all $i \in \mathbb{N}$ we have $0 < d(K_i \cap gK', x) < \epsilon$. For these i we define $u_i := u(K_i \cap gK', x)$, $p_i := p(K_i \cap gK', x)$, and we put $u := u(K \cap gK', x)$, $p := p(K \cap gK', x)$. We have $u_i \to u$, $p_i \to p$ for $i \to \infty$. Since K and gK' are in general relative position, there are uniquely determined $v, w \in S^{n-1}$ such that $(p, v) \in \eta \cap \text{Nor } K$, $(p, w) \in g(\eta' \cap \text{Nor } K')$ and $u \in\,]v, w[$. We have $p_i \in \text{bd } K_i \cap \text{bd } gK'$ for almost all i, since otherwise $(p_i, u_i) \in \text{Nor } K_i \cup \text{Nor } gK'$ for infinitely many i and therefore $(p, u) \in \text{Nor } K \cup \text{Nor } gK'$, hence $u \in \{v, w\}$. Since K_i and gK' do not touch each other we have $(p_i, u_i) \in \text{Nor } (K_i \cap gK') \cap ((\text{bd } K_i \cap \text{bd } gK') \times S^{n-1}) = \text{Nor } K_i \wedge \text{Nor } gK'$ for almost all i, hence for these i there are $v_i, w_i \in S^{n-1}$ with $(p_i, v_i) \in \text{Nor } K_i$, $(p_i, w_i) \in \text{Nor } gK'$ and $u_i \in [v_i, w_i]$. For almost all i we even have $u_i \notin \{v_i, w_i\}$, since otherwise again $u \in \{v, w\}$. As K and gK' are in general relative position, we can show the limit relations $v_i \to v$, $w_i \to w$ for $i \to \infty$ in the same way as the continuity of the maps π_1, π_2 defined in the proof of Lemma 5.1. Because of $(p, v) \in \eta$, $(p, w) \in g\eta'$ and since the sets $\eta, g\eta' \subset \Sigma$ are open, we have $(p_i, v_i) \in \eta$, $(p_i, w_i) \in g\eta'$ for almost all i, hence

$$x \in \liminf_{i\to\infty} M_\epsilon(K_i \cap gK', (\eta \cap \text{Nor } K_i) \sqcap g(\eta' \cap \text{Nor } K'))\,,$$

as asserted.

It follows that

$$\begin{aligned} &\mu_\epsilon\big(K \cap gK', (\eta \cap \text{Nor } K) \sqcap g(\eta' \cap \text{Nor } K')\big) \\ &\qquad = \lambda^n\big(M_\epsilon(K \cap gK', (\eta \cap \text{Nor } K) \sqcap g(\eta' \cap \text{Nor } K'))\big) \\ &\qquad \leq \lambda^n\big(\liminf_{i\to\infty} M_\epsilon(K_i \cap gK', (\eta \cap \text{Nor } K_i) \sqcap g(\eta' \cap \text{Nor } K'))\big) \\ &\qquad \leq \liminf_{i\to\infty} \mu_\epsilon\big(K_i \cap gK', (\eta \cap \text{Nor } K_i) \sqcap g(\eta' \cap \text{Nor } K')\big) \end{aligned}$$

for μ-a.e. g, and an application of Lemma 5.1 gives the asserted inequality.

It is not difficult to show that the inclusion

$$\begin{aligned} \limsup_{i\to\infty} M_\epsilon(K_i \cap gK', \text{Nor } K_i \wedge \text{Nor } gK') \quad &\subset \quad M_\epsilon(K \cap gK', \text{Nor } K \wedge \text{Nor } gK') \\ &\qquad \cup \text{ bd } (K \cap gK') \end{aligned}$$

holds if $g \in G_n$ is such that K and gK' do not touch each other. It follows on the one hand that for these g

$$\begin{aligned}
&\limsup_{i\to\infty} \mu_\epsilon(K_i \cap gK', \mathrm{Nor}\ K_i \wedge \mathrm{Nor}\ gK') \\
&\qquad = \limsup_{i\to\infty} \lambda^n\big(M_\epsilon(K_i \cap gK', \mathrm{Nor}\ K_i \wedge \mathrm{Nor}\ gK')\big) \\
&\qquad \le \lambda^n\big(\limsup_{i\to\infty} M_\epsilon(K_i \cap gK', \mathrm{Nor}\ K_i \wedge \mathrm{Nor}\ gK')\big) \\
&\qquad \le \lambda^n\big(M_\epsilon(K \cap gK', \mathrm{Nor}\ K \wedge \mathrm{Nor}\ gK')\big) \\
&\qquad = \mu_\epsilon(K \cap gK', \mathrm{Nor}\ K \wedge \mathrm{Nor}\ gK')\,.
\end{aligned}$$

On the other hand from the inequality shown in the first part of this proof it follows

$$\mu_\epsilon(K \cap gK', \mathrm{Nor}\ K \wedge \mathrm{Nor}\ gK') \le \liminf_{i\to\infty} \mu_\epsilon(K_i \cap gK', \mathrm{Nor}\ K_i \wedge \mathrm{Nor}\ gK')$$

for μ-a.e. $g \in G_n$. Now also the second assertion of Lemma 5.3 is established. ■

It is clear that results similar to Lemmas 5.2 and 5.3 are valid with the roles of the pairs (K, η) and (K', η') interchanged.

Lemma 5.4. *For all $\eta \in \mathcal{B}(\mathrm{Nor}\ K), \eta' \in \mathcal{B}(\mathrm{Nor}\ K')$ the map $g \mapsto \mu_\epsilon(K \cap gK', \eta \wedge g\eta')$ coincides with a Borel measurable map μ-almost everywhere.*

Proof: Let $g \in G_n$ such that K and gK' are in general relative position, and let $(g_i)_{i\in\mathbb{N}}$ be a sequence in G_n converging to g such that K and g_iK' are in general relative position for all $i \in \mathbb{N}$. If $\eta \subset \mathrm{Nor}\ K$ and $\eta' \subset \mathrm{Nor}\ K'$ are open subsets, then for every $\delta \in (0, \epsilon)$ the inclusion

$$M_\delta(K \cap gK', \eta \sqcap g\eta') \ \subset\ \liminf_{i\to\infty} M_\epsilon(K \cap g_iK, \eta \sqcap g_i\eta')$$

can be proved as in the first part of the proof of Lemma 5.3. Hence

$$\mu_\epsilon(K \cap gK', \eta \sqcap g\eta') \le \liminf_{i\to\infty} \mu_\epsilon(K \cap g_iK', \eta \sqcap g_i\eta')\,.$$

Now from Lemma 5.1 it follows that the map $g \mapsto \mu_\epsilon(K \cap gK', \eta \wedge g\eta')$ is well defined and lower semicontinuous on a subset of G_n of full measure for all open $\eta \subset \mathrm{Nor}\ K$ and $\eta' \subset \mathrm{Nor}\ K'$. Therefore the assertion is true for such η, η'. It is easy to show with the help of Lemma 5.2 that for open $\eta' \subset \mathrm{Nor}\ K'$ the set of all $\eta \in \mathcal{B}(\mathrm{Nor}\ K)$ for which the assertion is true is a Dynkin system. From this the assertion follows for $\eta \in \mathcal{B}(\mathrm{Nor}\ K)$ and open $\eta' \subset \mathrm{Nor}\ K'$. In a second step one can show the general assertion in an analogous manner. ■

Lemma 5.5. *There exist real numbers a_{kl}, $k, l \in \{0, \ldots, n-1\}$, depending only on ϵ, such that*

$$\int_{G_n} \mu_\epsilon(K \cap gK', \eta \wedge g\eta')\, d\mu(g) = \sum_{k,l=0}^{n-1} a_{kl} \Lambda_k(K, \eta) \Lambda_l(K', \eta')$$

for all $\eta \in \mathcal{B}(\mathrm{Nor}\ K), \eta' \in \mathcal{B}(\mathrm{Nor}\ K')$.

Proof: Let $P \in \mathcal{P}$ and $\eta \in \mathcal{B}(\Sigma)$. The map

$$\begin{aligned}
\psi : \mathcal{P} \times \mathcal{B}(\Sigma) \quad &\to \quad \mathbb{R} \\
(P', \eta') \quad &\mapsto \quad \int_{G_n} \mu_\epsilon\big(P \cap gP', (\eta \cap \mathrm{Nor}\ P) \wedge g(\eta' \cap \mathrm{Nor}\ P')\big)\, d\mu(g)\,,
\end{aligned}$$

which is well defined because of Lemma 5.4 and Theorem 2.1, satisfies the assumptions of Lemma 1.3: From Lemma 5.2 and the monotone convergence theorem it follows that the map $\psi(P',\cdot)$ is a measure for all $P' \in \mathcal{P}$. The left invariance of μ implies $\psi(gP', g\eta') = \psi(P',\eta')$ for all $P' \in \mathcal{P}$, $\eta' \in \mathcal{B}(\Sigma)$. For all $P_1', P_2' \in \mathcal{P}$ and all $\eta' \in \mathcal{B}(\Sigma)$ with $\eta' \cap \text{Nor } P_1' = \eta' \cap \text{Nor } P_2'$ we have

$$\begin{aligned} &[(\eta \cap \text{Nor } P) \wedge g(\eta' \cap \text{Nor } P_1')] \cap \text{Nor } (P \cap gP_1') \\ &\qquad = [(\eta \cap \text{Nor } P) \wedge g(\eta' \cap \text{Nor } P_1')] \cap \text{Nor } (P \cap gP_2') \\ &\qquad = [(\eta \cap \text{Nor } P) \wedge g(\eta' \cap \text{Nor } P_2')] \cap \text{Nor } (P \cap gP_2') \end{aligned}$$

for all $g \in G_n$, and since μ_ϵ is locally determined it follows $\psi(P_1',\eta') = \psi(P_2',\eta')$. Hence there exist numbers $c_0, \ldots, c_{n-1} \in \mathbb{R}$, depending only on ϵ, P and η, such that

$$\int_{G_n} \mu_\epsilon\big(P \cap gP', (\eta \cap \text{Nor } P) \wedge g(\eta' \cap \text{Nor } P')\big)\, d\mu(g) = \sum_{l=0}^{n-1} c_l \Lambda_l(P',\eta')\,.$$

The same argument can be applied to show that the coefficients c_l are linear combinations of the support measures of P:

$$c_l = \sum_{k=0}^{n-1} a_{kl} \Lambda_k(P,\eta)\,,$$

where the real numbers a_{kl} depend only on ϵ. Lemma 5.5 is now proved for polytopes. It can be extended to admissible pairs $K, K' \in \mathcal{K}$ in the same way as equation (4.5.19) on p. 247 in Schneider [5] is extended from strictly convex bodies to general convex bodies. Beside Lemmas 5.2, 5.3, Fatou's lemma and the dominated convergence theorem we here use the fact that, according to Theorem 2.1, a pair of convex bodies is admissible if one of the bodies is a polytope. ■

Proof of Theorem 3.1*:* Let $K, K' \in \mathcal{K}$ be admissible and let $\eta \in \mathcal{B}(\text{Nor } K)$, $\eta' \in \mathcal{B}(\text{Nor } K')$. According to equation (3) there are constants b_{jk} with

$$\int_{G_n} \Lambda_j(K \cap gK', \eta \wedge g\eta')\, d\mu(g) = \sum_{k=1}^{n} b_{jk} \int_{G_n} \mu_k(K \cap gK', \eta \wedge g\eta')\, d\mu(g)\,.$$

Now Lemma 5.5 shows that there are real constants c_{jkl} satisfying

$$\int_{G_n} \Lambda_j(K \cap gK', \eta \wedge g\eta')\, d\mu(g) = \sum_{k,l=0}^{n-1} c_{jkl} \Lambda_k(K,\eta) \Lambda_l(K',\eta')\,.$$

In the case $\eta = \text{Nor } K$, $\eta' = \text{Nor } K'$, this equation must be compatible with Theorem 1.1, thus we obtain for the constants $c_{jkl} = \alpha_{njk}$ if $l = n + j - k$ and $j + 1 \le k \le n - 1$, and $c_{jkl} = 0$ in all other cases. ■

In order to extend our results to the convex ring we need the following lemma. We use the notations introduced in connection with the inclusion-exclusion principle (4).

Lemma 5.6. *Let $K, K' \in \mathcal{R}$ be an admissible pair, let $K_1, \ldots, K_m$, $K_1', \ldots, K_{m'}' \in \mathcal{K}$ with $K = \cup_{i=1}^m K_i$, $K' = \cup_{i=1}^{m'} K_i'$ such that $K_v, K_{v'}'$ are admissible pairs of convex bodies for all $v \in S(m), v' \in S(m')$. Let further $\eta \in \mathcal{B}(\text{Nor } K)$, $\eta' \in \mathcal{B}(\text{Nor } K')$. Then for μ-a.e. $g \in G_n$ we have*

$$\Lambda_j(K_v \cap gK_{v'}', \eta \wedge g\eta') = \Lambda_j(K_v \cap gK_{v'}', (\eta \cap \text{Nor } K_v) \wedge g(\eta' \cap \text{Nor } K_{v'}'))$$

for all $j \in \{0, \ldots, n-1\}$ and all $v \in S(m)$, $v' \in S(m')$.

Proof: Let $v \in S(m)$, $v' \in S(m')$. For $g \in G_n$ we put

$$B_g := (\text{Nor } K_v \cup \text{Nor } gK'_{v'}) \cap ((\text{bd } K_v \cap \text{bd } gK'_{v'}) \times S^{n-1}).$$

As shown in the proof of Lemma 5.1 we have $\mu_\epsilon(K_v \cap gK'_{v'}, B_g) = 0$ for all $\epsilon > 0$ for μ-a.e. $g \in G_n$ and therefore

$$\Lambda_j(K_v \cap gK'_{v'}, B_g) = 0$$

for all $j \in \{0, \dots, n-1\}$ for μ-a.e. g. Now let $\eta \in \mathcal{B}(\text{Nor } K)$ and $\eta' \in \mathcal{B}(\text{Nor } K')$. For all $g \in G_n$ we have

$$(\eta \cap \text{Nor } K_v) \wedge g(\eta' \cap \text{Nor } K'_{v'}) \;\subset\; (\eta \wedge g\eta') \cap (\text{Nor } K_v \wedge \text{Nor } gK'_{v'}).$$

We show

$$((\eta \wedge g\eta') \cap (\text{Nor } K_v \wedge \text{Nor } gK'_{v'})) \backslash B_g \;\subset\; (\eta \cap \text{Nor } K_v) \wedge g(\eta' \cap \text{Nor } K'_{v'})$$

for all $g \in G_n$ such that K_w and $gK'_{w'}$ are in general relative position for all $w \in S(m)$, $w' \in S(m')$.

So let $g \in G_n$ such that K_w and $gK'_{w'}$ are in general relative position for all $w \in S(m)$, $w' \in S(m')$, and let $(x,u) \in ((\eta \wedge g\eta') \cap (\text{Nor } K_v \wedge \text{Nor } gK'_{v'})) \backslash B_g$. Then $(x,u) \notin \text{Nor } K_v \cup \text{Nor } gK'_{v'}$. According to the definition of "$\wedge$" there are $u_1, u_2, v_1, v_2 \in S^{n-1}$ with $(x,u_1) \in \eta$, $(x,u_2) \in \text{Nor } K_v$, $(x,v_1) \in g\eta'$, $(x,v_2) \in \text{Nor } gK'_{v'}$ and $u \in [u_1,v_1] \cap [u_2,v_2]$. We have $u \notin \{u_2, v_2\}$. If we let $w := \{i \in \{1,\dots,m\} : x \in K_i\} \in S(m)$ and $w' := \{i \in \{1,\dots,m'\} : x \in gK'_i\} \in S(m')$, then $v \subset w$ and $v' \subset w'$ and therefore $(x,u_2) \in \text{Nor } K_w$, $(x,v_2) \in \text{Nor } gK'_{w'}$. Because of the definition of the sets Nor K and Nor K' it follows from $\eta \subset \text{Nor } K$, $\eta' \subset \text{Nor } K'$ that $(x,u_1) \in \text{Nor } K_w$, $(x,v_1) \in \text{Nor } gK'_{w'}$. Since K_w and $gK'_{w'}$ are in general relative position, we infer $u_1 = u_2$ and $v_1 = v_2$. Hence (x,u) is included in $(\eta \cap \text{Nor } K_v) \wedge g(\eta' \cap \text{Nor } K'_{v'})$.

We now conclude that

$$\begin{aligned}
&\Lambda_j(K_v \cap gK'_{v'}, (\eta \cap \text{Nor } K_v) \wedge g(\eta' \cap \text{Nor } K'_{v'})) \\
&\qquad = \Lambda_j(K_v \cap gK'_{v'}, (\eta \wedge g\eta') \cap (\text{Nor } K_v \wedge \text{Nor } gK'_{v'}))
\end{aligned}$$

for all $j \in \{0,\dots,n-1\}$ for μ-a.e. $g \in G_n$. If K_v and $gK'_{v'}$ do not touch each other we have $\text{Nor } K_v \wedge \text{Nor } gK'_{v'} = \text{Nor } (K_v \cap gK'_{v'}) \cap ((\text{bd } K_v \cap \text{bd } gK'_{v'}) \times S^{n-1})$. Since $\Lambda_j(K_v \cap gK'_{v'}, \cdot)$ is concentrated on $\text{Nor } (K_v \cap gK'_{v'})$, the proof is finished if from $(x,u) \in (\eta \wedge g\eta') \cap \text{Nor } (K_v \cap gK'_{v'})$ it follows that $x \in \text{bd } K_v \cap \text{bd } gK'_{v'}$. We finally want to show this implication.

Let $(x,u) \in (\eta \wedge g\eta') \cap \text{Nor } (K_v \cap gK'_{v'})$. Then $x \in K_v$, and there is a $\bar{u} \in S^{n-1}$ with $(x,\bar{u}) \in \eta \subset \text{Nor } K$. We assume $x \in \text{int } K_v$. Then a fortiori $x \in \text{int } K$. We can choose a polytope $P_0 \in \mathcal{P}$ with $x \in \text{int } P_0$ and $P_0 \subset K$, and polytopes $P_1, \dots, P_k \in \mathcal{P}$ with $K \subset \cup_{i=0}^k P_i$ and $P_i \cap \text{int } P_0 = \emptyset$ for all $i \in \{1,\dots,k\}$. We define $K_{ij} := P_i \cap K_j$ for $i \in \{1,\dots,k\}$, $j \in \{1,\dots,m\}$. If we arrange the bodies P_0 and K_{ij} in a finite sequence $M_1, \dots, M_{km+1}$, then we have $K = \cup_{l=1}^{km+1} M_l$ and $(x,\bar{u}) \notin \text{Nor } M_w$ for all $w \in S(km+1)$ because of the choice of the polytopes $P_0, \dots, P_k$. It follows $(x,\bar{u}) \notin \text{Nor } K$ by the definition of the set Nor K. This contradiction shows $x \in \text{bd } K_v$, and the same argument proves $x \in \text{bd } gK_v$. ■

Proof of Theorem 3.3*:* Let $K, K' \in \mathcal{R}$ with $K = \cup_{i=1}^m K_i$, $K' = \cup_{i=1}^{m'} K'_i$, $K_1, \dots, K_m$, $K'_1, \dots, K'_{m'} \in \mathcal{K}$ and admissible pairs $K_v, K'_{v'}$ for all $v \in S(m), v' \in S(m')$. By the inclusion-exclusion principle we have for all $\eta \in \mathcal{B}(\text{Nor } K)$, $\eta' \in \mathcal{B}(\text{Nor } K')$, $g \in G_n$, $j \in \{0,\dots,n-1\}$

$$\Lambda_j(K \cap gK', \eta \wedge g\eta') = \sum_{v \in S(m)} \sum_{v' \in S(m')} (-1)^{|v|+|v'|} \Lambda_j(K_v \cap gK'_{v'}, \eta \wedge g\eta').$$

From Lemma 5.6 and Theorem 3.1 it now follows that

$$\begin{aligned}
&\int\limits_{G_n} \Lambda_j(K \cap gK', \eta \wedge g\eta')\, d\mu(g) \\
&\qquad = \sum_{v \in S(m)} \sum_{v' \in S(m')} (-1)^{|v|+|v'|} \int\limits_{G_n} \Lambda_j(K_v \cap gK'_{v'}, \eta \wedge g\eta')\, d\mu(g) \\
&\qquad = \sum_{v \in S(m)} \sum_{v' \in S(m')} (-1)^{|v|+|v'|} \\
&\qquad\qquad\qquad \times \int\limits_{G_n} \Lambda_j(K_v \cap gK'_{v'}, (\eta \cap \mathrm{Nor}\ K_v) \wedge g(\eta' \cap \mathrm{Nor}\ K'_{v'}))\, d\mu(g) \\
&\qquad = \sum_{v \in S(m)} \sum_{v' \in S(m')} (-1)^{|v|+|v'|} \sum_{k=j+1}^{n-1} \alpha_{njk} \Lambda_k(K_v, \eta \cap \mathrm{Nor}\ K_v) \Lambda_{n+j-k}(K'_{v'}, \eta' \cap \mathrm{Nor}\ K'_{v'}) \\
&\qquad = \sum_{v \in S(m)} \sum_{v' \in S(m')} (-1)^{|v|+|v'|} \sum_{k=j+1}^{n-1} \alpha_{njk} \Lambda_k(K_v, \eta) \Lambda_{n+j-k}(K'_{v'}, \eta') \\
&\qquad = \sum_{k=j+1}^{n-1} \alpha_{njk} \Lambda_k(K, \eta) \Lambda_{n+j-k}(K', \eta')\,.
\end{aligned}$$

■

The extension of the Crofton formula to the convex ring can be achieved in the same way. The following lemma is used, the proof of which is essentially the same as that of Lemma 5.6. It relies heavily on Theorem 2.2.

Lemma 5.7. *Let $K \in \mathcal{R}$, $K = K_1 \cup \cdots \cup K_m$ with $K_1, \ldots, K_m \in \mathcal{K}$, and let $\eta \in \mathcal{B}(\mathrm{Nor}\ K)$ and $q \in \{1, \ldots, n-1\}$. Then for μ_q-a.e. $E \in \mathcal{E}_q^n$ we have*

$$\Lambda_j(K_v \cap E, \eta \wedge E) = \Lambda_j(K_v \cap E, (\eta \cap \mathrm{Nor}\ K_v) \wedge E)$$

for all $j \in \{0, \ldots, q-1\}$ and all $v \in S(m)$.

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