

# Asymptotic estimates for best and stepwise approximation of convex bodies III\*

Stefan Glasauer and Peter M. Gruber

Source: Forum Mathematicum 9 (1997), 383 - 404

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**Abstract.** We consider approximations of a smooth convex body by inscribed and circumscribed convex polytopes as the number of vertices, resp. facets tends to infinity. The measure of deviation used is the difference of the mean width of the convex body and the approximating polytopes. The following results are obtained. (i) An asymptotic formula for best approximation. (ii) Upper and lower estimates for step-by-step approximation in terms of the so-called dispersion. (iii) For a sequence of best approximating inscribed polytopes the sequence of vertex sets is uniformly distributed in the boundary of the convex body where the density is specified explicitly.

## 1. Introduction and statement of results

**1.1** Let  $\mathcal{C}$  denote the space of *convex bodies* in Euclidean  $d$ -space  $\mathbb{E}^d$ , i.e. of all compact convex subsets of  $\mathbb{E}^d$  with non-empty interior. For notions not explained below we refer to [20]. Given  $C \in \mathcal{C}$  and  $k = 0, \dots, d$ , let  $W_k(C)$  be the  $k$ th quermassintegral of  $C$ .  $W_0 = V$  is the volume,  $dW_1$  the surface area and  $\frac{2}{\kappa_d}W_{d-1} = W$  the mean width. Here  $\kappa_d = V(B^d)$  denotes the volume of the solid unit ball  $B^d$  of  $\mathbb{E}^d$ . For convex bodies  $C, D$  with  $C \supset D$  we consider the following notion of deviation:

$$\delta^{W_k}(C, D) = W_k(C) - W_k(D).$$

If  $C \in \mathcal{C}$ , define  $\mathcal{P}_n^i = \mathcal{P}_n^i(C)$ ,  $n = d + 1, d + 2, \dots$ , to be the set of convex polytopes which have at most  $n$  vertices and are inscribed into  $C$ , that is, their vertices are on the boundary  $\text{bd}C$  of  $C$ . Similarly, let  $\mathcal{P}_{(n)}^c = \mathcal{P}_{(n)}^c(C)$  be the set of all convex polytopes circumscribed to  $C$  which have at most  $n$  facets, each touching  $C$ .

**1.2** It is well-known that for each  $C \in \mathcal{C}$  there is an  $\alpha > 0$  such that

$$(1.1) \quad \delta^V(C, \mathcal{P}_n^i) = \inf\{\delta^V(C, P) : P \in \mathcal{P}_n^i\} \leq \frac{\alpha}{n^{2/(d-1)}} \quad \text{for } n = d + 1, \dots$$

If  $C \in \mathcal{C}^2$ , i.e.  $\text{bd}C$  is a surface of class  $\mathcal{C}^2$ , and the Gauss curvature  $\kappa_C$  of  $\text{bd}C$  is positive, then

$$(1.2) \quad \delta^V(C, \mathcal{P}_n^i) \sim \frac{\text{del}_{d-1}}{2} \left( \int_{\text{bd}C} \kappa_C(x)^{1/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \quad \text{as } n \rightarrow \infty;$$

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\*AMS 1991 subject classification. Primary 11K06, 11K36, 41A60, 52A20, 52A25; Secondary 41A50, 52A40, 53B21, 54E52.

*Key words and phrases.* Best approximation, inscribed and circumscribed polytopes, mean width deviation, step-by-step approximation, uniform distribution.

see [2, 17] ( $d = 2$ ), [6] ( $d = 3$ ) and [10] (general  $d$ ). The constant  $\text{del}_{d-1} > 0$  depends only on  $d$  and  $\sigma$  is the ordinary surface area measure in  $\mathbb{E}^d$ . Results analogous to (1.1) and (1.2) hold also for  $\delta^V(C, \mathcal{P}_{(n)}^c)$ .

It is possible to extend (1.1) to all quermassintegrals: let  $C \in \mathcal{C}$ , then there is a constant  $\alpha > 0$  such that for each  $k = 0, \dots, d-1$ ,

$$(1.3) \quad \delta^{W_k}(C, \mathcal{P}_n^i) \leq \frac{\alpha}{n^{2/(d-1)}} \quad \text{for } n = d+1, \dots,$$

A similar result holds for  $\delta^{W_k}(C, \mathcal{P}_{(n)}^c)$ .

For the proof of the inequality (1.3), note that by a familiar result on Hausdorff approximation there exist a constant  $\alpha_1 > 0$  and polytopes  $P_n \in \mathcal{P}_n^i$ ,  $n = d+1, \dots$ , with

$$P_n \subset C \subset P_n + \varepsilon_n B^d \quad \text{for } n = d+1, \dots, \text{ and } \varepsilon_n = \frac{\alpha_1}{n^{2/(d-1)}},$$

see e.g. [8]. Here  $+$  denotes Minkowski addition. A version of Steiner's formula for parallel bodies together with the monotonicity of the quermassintegrals implies,

$$\begin{aligned} W_k(P_n) &\leq W_k(C) \leq W_k(P_n + \varepsilon_n B^d) \\ &= W_k(P_n) + \beta_1 \varepsilon_n W_{k+1}(P_n) + \dots + \beta_{d-k} \varepsilon_n^{d-k} W_d(P_n) \\ &\leq W_k(P_n) + \beta_1 \varepsilon_n W_{k+1}(C) + \dots + \beta_{d-k} \varepsilon_n^{d-k} W_d(C) \\ &\leq W_k(P_n) + \alpha_2 \varepsilon_n, \end{aligned}$$

where  $\beta_1, \dots, \beta_{d-k} > 0$  and  $\alpha_2 > 0$  depend on  $d$ , resp. on  $C$ . This gives (1.3), where  $\alpha = \alpha_1 \alpha_2$ .

More difficult is the extension of (1.2). We have succeeded only to show the following result.

**Theorem 1.** *Let  $C \in \mathcal{C} \cap \mathcal{C}^2$  with  $\kappa_C > 0$  be given. Then*

$$(1.4) \quad \delta^W(C, \mathcal{P}_n^i) \sim \frac{\text{div}_{d-1}}{d\kappa_d} \left( \int_{\text{bd}C} \kappa_C(x)^{d/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}},$$

$$(1.5) \quad \delta^W(C, \mathcal{P}_{(n)}^c) \sim \frac{\text{del}_{d-1}}{d\kappa_d} \left( \int_{\text{bd}C} \kappa_C(x)^{d/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}$$

as  $n \rightarrow \infty$ . Here  $\text{div}_{d-1}$  and  $\text{del}_{d-1}$  are positive constants depending only on  $d$ .

**Remark 1.** The case  $d = 2$  has been stated previously by L. Fejes Tóth [2], and proved by McClure and Vitale [17]. (In this case  $\delta^W$  coincides up to a multiplicative constant with the length deviation.) A related problem for curves in  $\mathbb{E}^d$  was treated by Gleason [4] whilst Enomoto [1] considered curves on Riemannian manifolds. For asymptotic stochastic approximation Müller [18] proved the following: let  $C \in \mathcal{C} \cap \mathcal{C}^3$  with  $\kappa_C > 0$  and let  $x_1, \dots, x_n$  be independent, identically distributed random points on  $\text{bd}C$  with density function  $f$ . Then for  $W(C) - E(W(\text{conv}\{x_1, \dots, x_n\}))$  there is an asymptotic formula as  $n \rightarrow \infty$ . If  $f$  is chosen appropriately, then it is of the form (1.4) but with a different constant. Here  $E$  and  $\text{conv}$  stand for mathematical expectation and convex hull, respectively.

**Remark 2.** The constants,  $\text{div}_k$ ,  $\text{del}_k$  were introduced in [10]. They are related to Dirichlet–Voronoi and to Delone tilings in  $\mathbb{E}^k$ . The first values are

$$\text{div}_1 = \frac{1}{12}, \text{div}_2 = \frac{5}{18\sqrt{3}}, \text{del}_1 = \frac{1}{6}, \text{del}_2 = \frac{1}{2\sqrt{3}};$$

see [2, 6, 7], but for  $k \geq 3$  it seems hopeless to determine the values explicitly.

**Remark 3.** We briefly sketch the first author’s original proof of Theorem 1, since we shall follow a different approach. For a convex body  $C \in \mathcal{C}$  with  $o \in \text{int}C$ , where  $\text{int}$  stands for interior, we denote by  $C^*$  its polar body, i.e.  $C^* = \{x \in \mathbb{E}^d : \langle x, y \rangle \leq 1 \text{ for all } y \in C\}$ . For  $C, D \in \mathcal{C}$  with  $o \in \text{int}C$  and  $C \subset D$ , let

$$\delta(C, D) := \int_{D \setminus C} \|x\|^{-d-1} d\lambda(x),$$

where  $\lambda$  is Lebesgue measure in  $\mathbb{E}^d$ . A simple computation using polar coordinates shows

$$(1.6) \quad \delta^W(C, D) = \frac{2}{d\kappa_d} \delta(C^*, D^*).$$

Now let  $C \in \mathcal{C} \cap \mathcal{C}^2$  with  $\kappa_C > 0$  and assume  $o \in \text{int}C$ . Then  $C^* \in \mathcal{C} \cap \mathcal{C}^2$  and  $\kappa_{C^*} > 0$  (cf. [20], p. 111). The relation

$$(1.7) \quad \delta(C^*, \mathcal{P}_n^i) \sim \frac{\text{del}_{d-1}}{2} \left( \int_{\text{bd}C^*} \kappa_{C^*}(x)^{1/(d+1)} \|x\|^{-d+1} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}$$

for  $n \rightarrow \infty$  can be established in the same way as the corresponding formula for  $\delta^V$  in [10]. By utilizing arguments of Kaltenbach [13], pp. 29 – 32, it can be seen that the integral on the right side equals

$$\int_{\text{bd}C} \kappa_C(x)^{d/(d+1)} d\sigma(x).$$

Now (1.5) follows from (1.6) and (1.7). In the same way also (1.4) can be established.

The proof of the second author was of a more direct type. It contained ideas of the proof of Theorem 1 in [10].

In this article we first prove a generalization of the crucial Lemma 3 in [10] to Riemannian manifolds, and then deduce (1.4) from it. Reasons for proceeding in this way are that the generalized Lemma is applicable also in other situations: it provides a short proof of the asymptotic formula for  $\delta^V(C, \mathcal{P}_{(n)}^c)$ , and it is used in the proof of Theorem 3 below. A similar but slightly more complicated proof can be given for (1.5); it makes use of a generalization of Lemma 2 in [10].

**Remark 4.** Among all convex bodies  $C \in \mathcal{C} \cap \mathcal{C}^2$  with  $\kappa_C > 0$  and of given mean width it is precisely the Euclidean balls which are asymptotically worst approximated in the sense of Theorem 1. This follows from Hölder’s inequality for integrals and the inequality  $W_1(C) \leq \kappa_d^{2-d} W_{d-1}(C)^{d-1}$  where equality holds precisely for balls; cf. [20], p.334, (6.4.7).

**Remark 5.** About refinements of Theorem 1 the same can be said as for Theorem 1 in [9] or [10], respectively. Comparing the asymptotic formulae (1.2) and (1.4), analogous results for other quermassintegrals seem plausible. In our attempts to prove such, two types of difficulties turn up: first, it seems to be complicated to relate in a simple but precise form the quermassintegrals of  $C$  and of its approximating polytopes; second, it is unclear how to compare the contributions of parts of  $\text{bd } C$  with different curvatures.

**1.3** Given a convex body  $C$ , the problem arises to determine polytopes in  $\mathcal{P}_n^i$  or  $\mathcal{P}_{(n)}^c$ ,  $n = d + 1, \dots$ , which approximate  $C$  reasonably well; see the discussion in [11] where we add the reference to Gordon, Meyer, and Reisner [5].

Let  $C \in \mathcal{C} \cap \mathcal{C}^2$  with  $\kappa_C > 0$  and let  $\gamma_C = \gamma_C(\cdot, \cdot)$  be the geodesic metric on  $\text{bd } C$  defined by the second fundamental form. Given a sequence  $x_1, x_2, \dots \in \text{bd } C$ , define its *dispersion*  $d_n(x_j) = d_n(\text{bd } C, \gamma_C, x_j)$ ,  $n = 1, 2, \dots$ , with respect to  $\gamma_C$  by

$$d_n(x_j) = \sup\{\inf\{\gamma_C(x, x_j), j = 1, \dots, n\} : x \in \text{bd } C\};$$

see [9, 10, 19]. In [10] it was proved that there are constants  $\beta, \gamma > 0$  depending on  $C$  such that for any sequence  $x_1, x_2, \dots \in \text{bd } C$  with  $d_n(x_j) \rightarrow 0$  as  $n \rightarrow \infty$ ,

$$(1.8) \quad \beta d_n(x_j)^{d+1} \leq \delta^V(C, \text{conv}\{x_1, \dots, x_n\}) \leq \gamma d_n(x_j)^2$$

for all sufficiently large  $n$ . The exponents  $d + 1$  and 2 are best possible. Further, there is a sequence  $y_1, y_2, \dots \in \text{bd } C$  with

$$d_n(y_j) \leq \frac{\delta}{n^{1/(d-1)}},$$

where  $\delta > 0$  depends only on  $C$ . Using a well-dispersed sequence in the unit cube in  $\mathbb{E}^{d-1}$  constructed by Niederreiter [19], this sequence can be given explicitly in a simple way. A result similar to (1.8) holds when  $\text{conv}\{x_1, \dots, x_n\}$  is replaced by  $H_C^+(x_1) \cap \dots \cap H_C^+(x_n)$  where  $H_C(x)$  is the supporting hyperplane of  $C$  at  $x$  and  $H_C^+(x)$  is the corresponding supporting halfspace.

The following result complements (1.8).

**Theorem 2.** *Let  $C \in \mathcal{C} \cap \mathcal{C}^2$  with  $\kappa_C > 0$ . Then there are constants  $\beta, \gamma, \delta > 0$ , depending on  $C$  such that the following hold:*

- (i) *For any sequence  $x_1, x_2, \dots \in \text{bd } C$  with  $d_n(x_j) = d_n(\text{bd } C, \gamma_C, x_j) \rightarrow 0$  as  $n \rightarrow \infty$  the inequalities*

$$(1.9) \quad \beta d_n(x_j)^{d+1} \leq \delta^W(C, \text{conv}\{x_1, \dots, x_n\}), \delta^W(C, H_C^+(x_1) \cap \dots \cap H_C^+(x_n)) \leq \gamma d_n(x_j)^2$$

*hold for all sufficiently large  $n$ . The exponents  $d + 1$  and 2 are best possible.*

- (ii) *There is an explicitly specifiable sequence  $y_1, y_2, \dots \in \text{bd } C$  with*

$$(1.10) \quad d_n(y_j) \leq \frac{\delta}{n^{1/(d-1)}} \text{ for } n = 1, 2, \dots$$

**Remark 6.** Theorem 2 shows that a suitable step-by-step approximation method is of the same order as best approximation. A sequence satisfying (1.10) is easy to construct: consider finitely many squares (i.e. cubes of dimension  $d - 1$ ) such that the interiors of their parallel projections on  $\text{bd } C$  cover  $\text{bd } C$ . In each square choose a well-dispersed sequence of points as described by Niederreiter [18]. Arrange the images of the points of these sequences in  $\text{bd } C$  into a sequence in the following way: take first the images of the first points of these sequences in a definite order, then take the images of all second points in the same order, then take the images of all third points, again in the same order, etc.

**Remark 7.** Similar results hold for all quermassintegrals, but the lower estimates are more difficult to prove than those in Theorem 2.

**1.4** While in general there is no hope to determine best approximating polytopes of a convex body with respect to any of the standard notions of deviation, some positive information has been obtained on the distribution of their vertices. For  $d = 2$  and for sufficiently differentiable  $C$  with  $\kappa_C > 0$ , McClure and Vitale [17] and, in a more precise form, Ludwig [15, 16] showed that for  $\delta^V$  and the Hausdorff metric  $\delta^H$  the vertices of the best approximating polygons are — in a definite sense — almost equally spaced on  $\text{bd } C$ . For general  $d$  Glasauer and Schneider [3] proved the following: given  $C \in \mathcal{C} \cap \mathcal{C}^2$  with  $\kappa_C > 0$ , let  $P_n \in \mathcal{P}_n^i, n = d + 1, \dots$ , be best approximating polytopes of  $C$  with respect to  $\delta^H$ . Then the sequence  $(\text{vert } P_n)$  of the sets of vertices of the  $P_n$ s is *uniformly distributed with respect to the density  $\kappa_C^{1/2}$* . That is, for each Jordan measurable set  $J \subset \text{bd } C$ ,

$$\frac{\#(J \cap \text{vert } P_n)}{n} \rightarrow \frac{\int_J \kappa_C(x)^{1/2} d\sigma(x)}{\int_{\text{bd } C} \kappa_C(x)^{1/2} d\sigma(x)} \quad \text{as } n \rightarrow \infty;$$

see [12, 14].  $J$  is called *Jordan measurable* if it is Borel and the  $\sigma$ -measure of its boundary in  $\text{bd } C$  is 0.  $\#$  means cardinal number.

A related result is the following.

**Theorem 3.** *Let  $C \in \mathcal{C} \cap \mathcal{C}^2$  with  $\kappa_C > 0$  and let  $P_n \in \mathcal{P}_n^i, Q_n \in \mathcal{P}_{(n)}^c, n = d + 1, \dots$ , be best approximating polytopes with respect to  $\delta^W$ . Then the sequences of sets  $(\text{vert } P_n)$  and  $(C \cap \text{bd } Q_n)$  are uniformly distributed in  $\text{bd } C$  with respect to the density  $\kappa_C^{d/(d+1)}$ .*

**Remark 7.** We give the proof only for  $(\text{vert } P_n)$ .

**Remark 8.** Our method of proof shows that a similar result holds for  $\delta^V$  with corresponding density  $\kappa_C^{1/(d+1)}$  and it seems plausible that there are also extensions to the other quermassintegrals.

## 2. Tools on tilings and coverings of manifolds

**2.1** In order to make the exposition more self-contained, we repeat the relevant parts of the definitions in [9].

Let  $M$  be a  $(d - 1)$ -dimensional (Riemannian) manifold of class  $\mathcal{C}^2$  with metric of class  $\mathcal{C}^0$ . Then for any  $p \in M$  there are a neighborhood  $U$  of  $p$  in  $M$  and a homeomorphism

$h = \text{"'"} of  $U$  onto an open ball  $U' = h(U)$  in  $\mathbb{E}^{d-1}$ . To any  $u \in U'$  there is assigned a quadratic form  $q_u(s) = q_{p,u}(s)$  on  $\mathbb{E}^{d-1}$ , the coefficients of which are continuous in  $u$ .$

A curve  $K$  in  $M$  is of class  $\mathcal{C}^1$  if it has a parametrization  $x : [\alpha, \beta] \rightarrow M$  such that for any  $U$  and corresponding  $h$  and any interval  $[\lambda, \mu] \subset [\alpha, \beta]$  with  $x([\lambda, \mu]) \subset U$  the function  $u = h \circ x : [\lambda, \mu] \rightarrow U'$  is of class  $\mathcal{C}^1$ . If  $K \subset U$ , its *length* is

$$(2.1) \quad \int_{\alpha}^{\beta} q_{u(\tau)}(\dot{u}(\tau))^{1/2} d\tau,$$

otherwise dissect  $K$  suitably and add. For  $x, y \in M$ , let  $\gamma_M(x, y)$  be the infimum of the lengths of the curves of class  $\mathcal{C}^1$  in  $M$  which connect  $x, y$ .  $\gamma_M$  is the *geodesic metric* on  $M$ . It determines the original topology on  $M$ . The *geodesic disc*  $D(x, \varrho)$  in  $M$  with center  $x \in M$  and radius  $\varrho > 0$  is the set  $\{y \in M : \gamma_M(x, y) \leq \varrho\}$ .

A set  $J$  in  $M$  is *Jordan measurable* if its closure  $\text{cl}J$  is compact and for any  $p, U, h$  and any neighbourhood  $V$  of  $p$  with  $\text{cl}V \subset U$  for which  $V'$  is Jordan measurable in  $\mathbb{E}^{d-1}$ , also  $(J \cap V)'$  is Jordan measurable in  $\mathbb{E}^{d-1}$ . If  $M$  is compact, then it is Jordan measurable and so are geodesic discs. Finite unions, intersections and set differences of Jordan measurable sets are again Jordan measurable. If  $J \subset V$ , then

$$(2.2) \quad \omega(J) = \int_{J'} (\det q_u)^{1/2} du \quad (du = du^1 \dots du^{d-1})$$

is its *Jordan measure*; otherwise dissect  $J$  suitably and add. Clearly,  $\omega$  gives rise to a (Riemann) integral on  $M$ .  $\omega$  may be extended to a Borel measure on  $M$ . Using this measure, one may alternatively define  $J \subset M$  to be Jordan measurable if  $\text{cl}J$  is compact and  $\omega(\text{bd } J) = 0$ ; compare subsection 1.4.

Let  $|\cdot|$  and  $\|\cdot\|$  denote volume and Euclidean norm in  $\mathbb{E}^{d-1}$ .

**2.2** Our main tool is the following result which extends Lemma 3 in [10].

**Lemma 1.** *Let  $J \subset M$  be Jordan measurable and  $f : M \rightarrow \mathbb{R}^+$  continuous and bounded. Then*

$$(2.3) \quad \inf_J \left\{ \int_J \min\{\gamma_M(v, x)^2 : v \in D\} f(x) d\omega(x) : D \subset M, \#D \leq n \right\} \\ \sim \text{div}_{d-1} \left( \int_J f(x)^{(d-1)/(d+1)} d\omega(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}$$

as  $n \rightarrow \infty$ , where  $\text{div}_{d-1} > 0$  is a constant depending only on  $d$ .

**Proof.** Let  $\lambda > 1$ .

1. We need Lemma 3 in [10] in the following form.

(2.4) Let  $Q = \{s \in \mathbb{E}^{d-1} : 0 \leq s^i \leq 1\}$ . For every  $k = 1, 2, \dots$ , there is a finite set  $D \subset \mathbb{E}^{d-1}$  with  $\#D \leq k$  for which

$$w_k = \int_Q \min\{\|r - s\|^2 : r \in D\} ds \quad (ds = ds^1 \dots ds^{d-1})$$

is minimal. Further, there is a constant,  $\text{div}_{d-1} > 0$ , depending only on  $d$ , such that

$$w_k \sim \frac{\text{div}_{d-1}}{k^{2/(d-1)}} \quad \text{as } k \rightarrow +\infty.$$

If a point  $r \in D$  is not in  $Q$ , then the point  $q \in Q$  closest to it satisfies the inequality  $\|r - s\| > \|q - s\|$  for all  $s \in Q$ . Hence we may suppose that the set  $D$  in (2.4) is contained in  $Q$ . This remark together with a suitable linear transformation leads to the following version of (2.4).

(2.5) Let  $q(\cdot)$  be a positive quadratic form on  $\mathbb{E}^{d-1}$  and  $R \subset \mathbb{E}^{d-1}$  a square in the sense of the norm  $q^{1/2}$ . Then for all sufficiently large  $k$  the following hold.

(i) For any set  $D \subset \mathbb{E}^{d-1}$  with  $\#D \leq k$ ,

$$\int_R \min\{q(r - s) : r \in D\} ds \geq \frac{\text{div}_{d-1}}{\lambda} |R|^{(d+1)/(d-1)} (\det q)^{1/(d-1)} \frac{1}{k^{2/(d-1)}}.$$

(ii) There is a set  $E \subset R$  with  $\#E \leq k$  such that

$$\int_R \min\{q(r - s) : r \in E\} ds \leq \lambda \text{div}_{d-1} |R|^{(d+1)/(d-1)} (\det q)^{1/(d-1)} \frac{1}{k^{2/(d-1)}}.$$

A further needed tool is the following mean inequality.

$$(2.6) \quad \left( \frac{\sigma_1^{(d+1)/(d-1)} + \dots + \sigma_n^{(d+1)/(d-1)}}{n} \right)^{(d-1)/(d+1)} \geq \frac{\sigma_1 + \dots + \sigma_n}{n} \quad \text{for } \sigma_1, \dots, \sigma_n \geq 0.$$

2. For  $p \in M$ , let  $U, h = \text{"'"}, q_{(\cdot)}$  and  $U' = h(U)$  be explained as above.  $q_{(\cdot)} = q_{p,(\cdot)}$  is positive and its coefficients are continuous in  $U'$ . Further,  $f$  is continuous and positive on  $U$ . Hence, by replacing  $U$  by a suitable smaller neighbourhood of  $p$  and changing notation if necessary, we may assume the following, where  $q_u = q_{p,u}$ ,  $q_{p'}$  and  $f_{p'} = f(p)$ :

$$\begin{aligned} (1/\lambda)q_{p'}(s) &\leq q_u(s) \leq \lambda q_{p'}(s) && \text{for } u \in U', s \in \mathbb{E}^{d-1}, \\ (1/\lambda)(\det q_{p'})^{1/2} &\leq (\det q_u)^{1/2} \leq \lambda(\det q_{p'})^{1/2} && \text{for } u \in U', \\ (1/\lambda)f_{p'} &\leq f(x) \leq \lambda f_{p'} && \text{for } x \in U. \end{aligned}$$

We now show,

$$(1/\lambda)q_{p'}(x' - y') \leq \gamma_M(x, y)^2 \leq \lambda q_{p'}(x' - y') \quad \text{for } x, y \in U, \gamma_M(x, y) < \text{dist}(x, \text{bd } U),$$

where  $\text{dist}$  means distance of sets in  $M$  in the sense of the metric  $\gamma_M$ .

To see the first inequality, consider all curves of class  $\mathcal{C}^1$  connecting  $x, y$  with length  $< \text{dist}(x, \text{bd } U)$ . These curves are all contained in  $U$ . Now apply the definition (2.1) of length, the above inequality between  $q_p$  and  $q_u$  and the definition of  $\gamma_M$  to these curves. For the second inequality connect  $x', y'$  with a line segment  $S$  and consider the curve  $h^{-1}(S) \subset U$ . Then proceed as in the case of the first inequality.

In each neighbourhood  $U$  choose an open Jordan measurable neighbourhood  $V$  of  $p$  with  $\text{cl}V \subset U$ . As  $p$  ranges over  $M$ , the  $V$ s form an open covering of the compact set  $\text{cl}J$ . Let  $V_l, l = 1, \dots, m$ , be a finite subcover. Then the sets

$$I_l = J \cap (V_l \setminus (V_1 \cup \dots \cup V_{l-1})), \quad l = 1, \dots, m,$$

are pairwise disjoint Jordan measurable sets with union  $J$ . We clearly may assume that  $J$  is open.

Thus, after omitting the empty ones among  $I_1, \dots, I_m$ , renumbering and changing notation, if necessary, we obtain the following.

(2.7) There are points  $p_l \in M$ ,  $l = 1, \dots, m$ , with corresponding  $U_l, V_l, h_l = \text{"'"}'$ ,  $q_u = q_{p_l, u}$ ,  $q_l = q_{p_l, p'_l}$  and  $f_l = f(p_l)$  such that

- (i)  $(1/\lambda)q_l(s) \leq q_u(s) \leq \lambda q_l(s)$  for  $u \in U'_l, s \in \mathbb{E}^{d-1}$ ,
- (ii)  $(1/\lambda)(\det q_l)^{1/2} \leq (\det q_u)^{1/2} \leq \lambda(\det q_l)^{1/2}$  for  $u \in U'_l$ ,
- (iii)  $(1/\lambda)f_l \leq f(x) \leq \lambda f_l$  for  $x \in U_l$ ,
- (iv)  $(1/\lambda)q_l(x' - y') \leq \gamma_M(x, y)^2 \leq \lambda q_l(x' - y')$  for  $x, y \in U_l$   
with  $\gamma_M(x, y) < \text{dist}(x, \text{bd } U_l)$ ,
- (v)  $J = I_1 \cup \dots \cup I_m$  where the  $I_l$ s ( $\subset V_l$ ) are Jordan measurable, pairwise disjoint, and have non-empty interior.

3. Next the following lower estimate will be established.

(2.8) For all sufficiently large  $n$  holds:

$$\begin{aligned} & \inf \left\{ \int_J \min \{ \gamma_M(x, y)^2 : v \in D \} f(x) d\omega(x) : D \subset M, \#D \leq n \right\} \\ & \geq \frac{\text{div}_{d-1}}{\lambda^{12}} \left( \int_J f(x)^{(d-1)/(d+1)} d\omega(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}. \end{aligned}$$

For each  $l = 1, \dots, m$ , choose sets  $S_{li}, i = 1, \dots, i_l$ , such that

(2.9) the sets  $S_{li}$  are compact, pairwise disjoint sets in the interior  $\text{int}I_l$  of  $I_l$  where the images  $S'_{li}$  are squares in  $\mathbb{E}^{d-1}$  in the sense of the norm  $q_l^{1/2}$  and

$$\sum_i |S'_{li}| \geq \frac{1}{\lambda} |I'_l|.$$



The  $S_{li}$ s are compact and contained in the open set  $U_l$ . By (2.7v) the  $I_l$ s are pairwise disjoint. Hence there is a  $\delta > 0$  such that

$$(2.10) \quad \text{dist}(S_{li}, \text{bd } U_l) > \delta \quad \text{for each pair } (l, i),$$

$$(2.11) \quad \text{dist}(S_{li}, S_{kj}) > 2\delta \quad \text{for all pairs } (l, i) \neq (k, j).$$

For each  $n = 1, 2, \dots$ , choose a set  $D_n \subset M$  with  $\#D_n \leq n$ , for which

$$(2.12) \quad \inf \left\{ \int_J \min\{\gamma_M(v, x)^2 : v \in D\} f(x) d\omega(x) : D \subset M, \#D \leq n \right\} \\ \geq \frac{1}{\lambda} \int_J \min\{\gamma_M(v, x)^2 : v \in D_n\} f(x) d\omega(x).$$

$f$  is bounded and there are sets  $D \subset J$  with  $\#D \leq n$  which are arbitrarily densely distributed in  $J$  if  $n$  is sufficiently large. This implies that the left hand side and thus also the right hand side in (2.12) tends to 0 as  $n \rightarrow \infty$ . Since  $f$  is positive, the latter implies in turn that the sets  $D_n$  are arbitrarily densely distributed in  $\text{int} J$  if  $n$  is sufficiently large. Thus, in particular, the following hold: if

$$D_{nli} = \{v \in D_n : \text{dist}(v, S_{li}) < \delta\}, k_{nli} = \#D_{nli},$$

then

$$(2.13) \quad k_{nli} \rightarrow +\infty \text{ as } n \rightarrow \infty,$$

$$(2.14) \quad \text{if } n \text{ is sufficiently large, then } \min\{\gamma_M(v, x)^2 : v \in D_n\} = \min\{\gamma(v, x)^2 : v \in D_{nli}\} \\ \text{for each } x \in S_{li}.$$

By (2.11) the sets  $D_{nli}$  are pairwise disjoint for each  $n$ . This implies that

$$(2.15) \quad k_{n11} + \dots + k_{n1i_1} + \dots + k_{nmi_m} \leq n.$$

The definition of  $\omega$  in (2.2) together with propositions (2.14), (2.10), (2.7), (2.13) and (2.5) yields the following:

(2.16) if  $n$  is sufficiently large, then we have

$$\int_{S_{li}} \min\{\gamma_M(v, x)^2 : v \in D_n\} f(x) d\omega(x) \\ = \int_{S_{li}} \min\{\gamma_M(v, x)^2 : v \in D_{nli}\} f(x) d\omega(x) \\ \geq \frac{1}{\lambda^3} \int_{S'_{li}} \min\{q_l(r - s) : r \in D'_{nli}\} f_l(\det q_l)^{1/2} ds \\ \geq \frac{\text{div}_{d-1}}{\lambda^4} |S'_{li}|^{(d+1)/(d-1)} (\det q_l)^{(d+1)/2(d-1)} f_l \frac{1}{k_{nli}^{2/(d-1)}}.$$

Now, using (2.12), (2.7), (2.9), (2.16), (2.6), (2.9), (2.15), (2.7), (2.2) and (2.7) again, we see that for all sufficiently large  $n$ ,

$$\begin{aligned}
& \inf \left\{ \int_J \min \{ \gamma_M(v, x)^2 : v \in D \} f(x) d\omega(x) : D \subset M, \#D \leq n \right\} \\
& \geq \frac{1}{\lambda} \int_J \min \{ \gamma_M(v, x)^2 : v \in D_n \} f(x) d\omega(x) \\
& \geq \frac{1}{\lambda} \sum_{l,i} \int_{\tilde{S}_{li}} \min \{ \gamma_M(v, x)^2 : v \in D_{nli} \} f(x) d\omega(x) \\
& \geq \frac{\text{div}_{d-1}}{\lambda^5} \sum_{l,i} |S'_{li}|^{(d+1)/(d-1)} (\det q_l)^{(d+1)/2(d-1)} f_l \frac{1}{k_{nli}^{2/(d-1)}} \\
& = \frac{\text{div}_{d-1}}{\lambda^5} \sum_{l,i} (|S'_{li}| (\det q_l)^{1/2} f_l^{(d-1)/(d+1)} \frac{1}{k_{nli}})^{(d+1)/(d-1)} k_{nli} \\
& \geq \frac{\text{div}_{d-1}}{\lambda^5} \left( \sum_{l,i} |S'_{li}| (\det q_l)^{1/2} f_l^{(d-1)/(d+1)} \frac{1}{k_{nli}} k_{nli} \right)^{(d+1)/(d-1)} \frac{1}{(k_{n11} + \dots + k_{nmi_m})^{2/(d-1)}} \\
& \geq \frac{\text{div}_{d-1}}{\lambda^8} \left( \sum_l |I'_l| (\det q_l)^{1/2} f_l^{(d-1)/(d+1)} \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \\
& = \frac{\text{div}_{d-1}}{\lambda^8} \left( \sum_l \int_{I'_l} f_l^{(d-1)/(d+1)} (\det q_l)^{1/2} ds \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \\
& \geq \frac{\text{div}_{d-1}}{\lambda^{12}} \left( \sum_l \int_{I_l} f(x)^{(d-1)/(d+1)} d\omega(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \\
& = \frac{\text{div}_{d-1}}{\lambda^{12}} \left( \int_J f(x)^{(d-1)/(d+1)} d\omega(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}},
\end{aligned}$$

concluding the proof of (2.8).

4. Finally, the corresponding upper estimate will be proved.

(2.17) For all sufficiently large  $n$ , the following inequality holds:

$$\begin{aligned}
& \inf \left\{ \int_J \min \{ \gamma_M(v, x)^2 : v \in D \} f(x) d\omega(x) : D \subset M, \#D \leq n \right\} \\
& \leq \lambda^{14} \text{div}_{d-1} \left( \int_J f(x)^{(d-1)/(d+1)} d\omega(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}.
\end{aligned}$$

For each  $l = 1, \dots, m$ , choose in  $\mathbb{E}^{d-1}$  a facet-to-facet tiling with closed squares in the sense of the norm  $q_l^{1/2}$ , each having area

$$(2.18) \quad \frac{\sum_l |I'_l| (\det q_l)^{1/2} f_l^{(d-1)/(d+1)}}{(\det q_l)^{1/2} f_l^{(d-1)/(d+1)}} \alpha,$$

where  $\alpha > 0$  is chosen such that the following relations hold: let  $T'_{i_l}, i = 1, \dots, i_l$ , be the squares in  $\mathbb{E}^{d-1}$  that meet  $I'_l$ , then

$$(2.19) \quad I'_l \subset T'_{l1} \cup \dots \cup T'_{li} \subset U'_l,$$

$$(2.20) \quad |I'_l| \leq i_l |T'_{l1}| \leq \lambda |I'_l|.$$

Clearly,

$$(2.21) \quad T_{li} = h_l^{-1}(T'_{li}) \subset M \text{ is Jordan measurable.}$$

We now show that

$$(2.22) \quad \text{for } i_0 = i_1 + \dots + i_m \text{ the inequality}$$

$$\frac{i_0}{i_l} \leq \lambda \frac{\sum_l |I'_l| (\det q_l)^{1/2} f_l^{(d-1)/(d+1)}}{|I'_l| (\det q_l)^{1/2} f_l^{(d-1)/(d+1)}}$$

holds.

For the proof take the right hand side inequality in (2.20), multiply both sides by  $(\det q_l)^{1/2} f_l^{(d-1)/(d+1)}$ , insert the value for  $|T'_{li}|$  from (2.18) and sum over  $l$ . This gives  $i_0 \alpha \leq \lambda$  or  $\alpha \leq \lambda/i_0$ . Now consider the left hand side inequality in (2.20), insert the value (2.18) for  $|T'_{li}|$ , rearrange and note that  $\alpha \leq \lambda/i_0$ . The proof of (2.22) is now complete.

Let  $k_0$  be so large that the conclusion of (2.5) holds for all  $k \geq k_0$ . Then

$$(2.23) \quad \text{for each } k \geq k_0 \text{ there is a set } E_{kli} \subset T_{li} \text{ with } \#E_{kli} \leq k \text{ and}$$

$$\begin{aligned} & \int_{T'_{li}} \min\{q_l(r-s) : r \in E'_{kli}\} f_l (\det q_l)^{1/2} ds \\ & \leq \lambda \operatorname{div}_{d-1} |T'_{li}|^{(d+1)/(d-1)} (\det q_l)^{(d+1)/2(d-1)} f_l \frac{1}{k^{2/(d-1)}}. \end{aligned}$$

For the proof of (2.17) we first consider the case  $n = i_0 k$ ,  $k \geq k_0$  where  $i_0 = i_1 + \dots + i_m$ . Let

$$E_n = \bigcup_{l,i} E_{kli}.$$

Clearly,  $\#E_n \leq i_0 k = n$ . This combined with (2.7), (2.19), (2.21), (2.2), (2.7), (2.23), (2.20), (2.22), (2.7), (2.2) and (2.7) then shows that

$$(2.24) \quad \inf_J \left\{ \int \min\{\gamma_M(v, x)^2 : v \in D\} f(x) d\omega(x) : D \subset M, \#D \leq n \right\}$$

$$\begin{aligned}
&\leq \int_J \min\{\gamma_M(v, x)^2 : v \in E_n\} f(x) d\omega(x) \\
&\leq \sum_{l,i} \int_{T_{li}} \min\{\gamma_M(v, x)^2 : v \in E_{kli}\} f(x) d\omega(x) \\
&\leq \lambda^3 \sum_{l,i} \int_{T'_{li}} \min\{q_l(r-s) : r \in E'_{kli}\} f_l(\det q_l)^{1/2} ds \\
&\leq \lambda^4 \operatorname{div}_{d-1} \sum_{l,i} |T'_{li}|^{(d+1)/(d-1)} (\det q_l)^{(d+1)/2(d-1)} f_l \frac{1}{k^{2/(d-1)}} \\
&= \lambda^4 \operatorname{div}_{d-1} \sum_l (i_l |T'_{l1}|)^{(d+1)/(d-1)} (\det q_l)^{(d+1)/2(d-1)} f_l \frac{1}{(i_l k)^{2/(d-1)}} \\
&\leq \lambda^7 \operatorname{div}_{d-1} \sum_l |I'_l|^{(d+1)/(d-1)} (\det q_l)^{(d+1)/2(d-1)} f_l \left(\frac{i_0}{i_l}\right)^{2/(d-1)} \frac{1}{(i_0 k)^{2/(d-1)}} \\
&\leq \lambda^9 \operatorname{div}_{d-1} \sum_l |I'_l|^{(d+1)/(d-1)} (\det q_l)^{(d+1)/2(d-1)} f_l \times \\
&\quad \times \frac{(\sum_l |I'_l| (\det q_l)^{1/2} f_l^{(d-1)/(d+1)})^{2/(d-1)}}{(|I'_l| (\det q_l)^{1/2} f_l^{(d-1)/(d+1)})^{2/(d-1)}} \frac{1}{(i_0 k)^{2/(d-1)}} \\
&= \lambda^9 \operatorname{div}_{d-1} \left( \sum_l |I'_l| (\det q_l)^{1/2} f_l^{(d-1)/(d+1)} \right)^{(d+1)/(d-1)} \frac{1}{(i_0 k)^{2/(d-1)}} \\
&= \lambda^9 \operatorname{div}_{d-1} \left( \sum_l \int_{I'_l} f_l^{(d-1)/(d+1)} (\det q_l)^{1/2} ds \right)^{(d+1)/(d-1)} \frac{1}{(i_0 k)^{2/(d-1)}} \\
&\leq \lambda^{13} \operatorname{div}_{d-1} \left( \sum_l \int_{I_l} f(x)^{(d-1)/(d+1)} d\omega(x) \right)^{(d+1)/(d-1)} \frac{1}{(i_0 k)^{2/(d-1)}} \\
&= \lambda^{13} \operatorname{div}_{d-1} \left( \int_J f(x)^{(d-1)/(d+1)} d\omega(x) \right)^{(d+1)/(d-1)} \frac{1}{(i_0 k)^{2/(d-1)}}.
\end{aligned}$$

This concludes the proof of (2.17) for  $n$  of the form  $i_0 k$ .

For general  $n$  we argue as follows: let  $k_1 \geq k_0$  be so large that

$$(2.25) \quad \left(\frac{k+1}{k}\right)^{2/(d-1)} \leq \lambda \text{ for all } k \geq k_1.$$

Now, let  $n \geq i_0 k_1$  and choose  $k \geq k_1$  with  $i_0 k \leq n < i_0(k+1)$ . Then we obtain from (2.24) and (2.25) that

$$\begin{aligned}
&\inf\left\{ \int_J \min\{\gamma_M(v, x)^2 : v \in D\} f(x) d\omega(x) : D \subset M, \#D \leq n \right\} \\
&\leq \inf\left\{ \int_J \min\{\gamma_M(v, x)^2 : v \in D\} f(x) d\omega(x) : D \subset M, \#D \leq i_0 k \right\} \\
&\leq \lambda^{13+1} \operatorname{div}_{d-1} \left( \int_J f(x)^{(d-1)/(d+1)} d\omega(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}},
\end{aligned}$$

concluding the proof of (2.17) for general  $n$ .

5. Since  $\lambda > 1$  was arbitrary, Lemma 1 is an immediate consequence of propositions (2.8) and (2.17).

**2.3** The proof of the following result is left to the reader.

**Lemma 2.** *Let  $J \subset M$  be Jordan measurable and let  $\mu > 1$ . Then there is a constant  $\tau > 0$  such that*

$$\frac{1}{\mu} \kappa_{d-1} \varrho^{d-1} \leq \omega(D(x, \varrho)) \leq \mu \kappa_{d-1} \varrho^{d-1} \text{ for all } x \in J \text{ and } 0 < \varrho \leq \tau.$$

**2.4** The definition of dispersion in 1.3 may easily be generalized to any compact metric space and thus in particular to the closure of a Jordan measurable set in  $M$  endowed with the geodesic metric  $\gamma_M$ . Essentially the same proof as for Lemma 2 in [9] then yields the next proposition.

**Lemma 3.** *Let  $J \subset M$  be Jordan measurable. Then there are constants  $\alpha, \delta > 0$  such that the following hold:*

(i) *For any sequence  $x_1, \dots \in \text{cl}J$  with  $d_n(x_j) = d_n(\text{cl}J, \gamma_M, x_j) \rightarrow 0$  as  $n \rightarrow \infty$ ,*

$$d_n(x_j) > \frac{\alpha}{n^{1/(d-1)}}$$

*for all sufficiently large  $n$ .*

(ii) *There is an explicitly specifiable sequence  $y_1, \dots \in \text{cl}J$  such that*

$$d_n(y_j) \leq \frac{\delta}{n^{1/(d-1)}} \text{ for } n = 1, 2, \dots$$

### 3. Proof of Theorem 1

We prove only (1.4). Let  $\lambda > 1$ .

**3.1** Let  $\text{bd}C$  be endowed with the second fundamental form and let  $\gamma_C$  and  $\omega$  be the corresponding geodesic metric and surface area measure, respectively; see 2.1.

$\omega$  and the ordinary surface area measure  $\sigma$  (which corresponds to the first fundamental form) are related by the Radon–Nikodym derivative

$$(3.1) \quad \frac{d\omega}{d\sigma} = \kappa_C^{1/2}.$$

Now we refer to the outline of the proof of Theorem 3 in [9]. Corresponding to  $\lambda$  there are open neighbourhoods  $U_l, l = 1, \dots, m$ , in  $\text{bd}C$  which cover  $\text{bd}C$ . Since  $\text{bd}C$  is compact, Lebesgue’s theorem shows that there is a  $\delta > 0$  such that for any  $v \in \text{bd}C$  the geodesic disc with center  $v$  and radius  $\delta$  is contained in one of the neighbourhoods  $U_l$ . Thus proposition (5.1) in [9] shows the following:

$$(3.2) \quad \frac{\gamma_C(v, x)^2}{2\lambda^5} \leq \text{dist}(v, H_C(x)) \leq \frac{\lambda^5 \gamma_C(v, x)^2}{2} \text{ for } v, x \in \text{bd } C \text{ with } \gamma_C(v, x) < \delta.$$

Here  $\text{dist}$  means the ordinary distance of sets in  $\mathbb{E}^d$ .

**3.2** Let  $P_n \in \mathcal{P}_n^i$  be best approximating polytopes of  $C$  with respect to  $\delta^W$ . Then we have

$$\delta^W(C, \mathcal{P}_n^i) = \delta^W(C, P_n) = \frac{2}{d\kappa_d} \int_{\text{bd } C} \min\{\text{dist}(v, H_C(x)) : v \in \text{vert } P_n\} \kappa_C(x) d\sigma(x).$$

Clearly,  $\delta^W(C, P_n) \rightarrow 0$  as  $n \rightarrow \infty$  and since  $C$  is strictly convex, the following holds for all sufficiently large  $n$ : for any  $x \in \text{bd } C$  there is a  $v \in \text{vert } P_n$  with  $\gamma_C(v, x) < \delta$  and  $\text{dist}(v, H_C(x)) = \min\{\text{dist}(w, H_C(x)) : w \in \text{vert } P_n\}$ . Combined with (3.2) this shows that

(3.3) for all sufficiently large  $n$  we obtain

$$\begin{aligned} \delta^W(C, \mathcal{P}_n^i) &\geq \frac{1}{\lambda^5 d\kappa_d} \int_{\text{bd } C} \min\{\gamma_C(v, x)^2 : v \in \text{vert } P_n\} \kappa_C(x) d\sigma(x) \\ &\geq \frac{1}{\lambda^5 d\kappa_d} \inf\left\{ \int_{\text{bd } C} \min\{\gamma_C(v, x)^2 : v \in D\} \kappa_C(x) d\sigma(x) : D \subset \text{bd } C, \#D \leq n \right\}. \end{aligned}$$

Since  $\delta^W(C, \mathcal{P}_n^i) = \delta^W(C, P_n) \rightarrow 0$ , also the last expression in (3.3) tends to 0 as  $n \rightarrow \infty$ . Noting that  $\kappa_C > 0$  we thus obtain the following: let  $D_n \subset \text{bd } C$  with  $\#D_n \leq n$  be chosen such that the infimum in this expression is attained for  $D = D_n$ , then for all sufficiently large  $n$  we have the following: for any  $x \in \text{bd } C$  there is a  $v \in D_n$  with  $\gamma_C(v, x) < \delta$ . Hence (3.2) implies that

(3.4) for all sufficiently large  $n$  we have:

$$\begin{aligned} &\frac{\lambda^5}{d\kappa_d} \inf\left\{ \int_{\text{bd } C} \min\{\gamma_C(v, x)^2 : v \in D\} \kappa_C(x) d\sigma(x) : D \subset \text{bd } C, \#D \leq n \right\} \\ &\geq \frac{2}{d\kappa_d} \int_{\text{bd } C} \min\{\text{dist}(v, H_C(x)) : v \in \text{vert conv } D_n\} \kappa_C(x) d\sigma(x) \\ &\geq \delta^W(C, \mathcal{P}_n^i). \end{aligned}$$

**3.3** Combining (3.3), (3.4), (3.1), Lemma 1 and again (3.1) we obtain the following:

(3.5) for all sufficiently large  $n$  holds:

$$\begin{aligned} &\delta^W(C, \mathcal{P}_n^i) \\ &\left\{ \begin{array}{l} \leq \\ \geq \end{array} \right\} \left\{ \begin{array}{l} \lambda^5 \\ \frac{1}{\lambda^5} \end{array} \right\} \times \frac{1}{d\kappa_d} \inf\left\{ \int_{\text{bd } C} \min\{\gamma_C(v, x)^2 : v \in D\} \kappa_C(x) d\sigma(x) : D \subset \text{bd } C, \#D \leq n \right\} \\ &\left\{ \begin{array}{l} \leq \\ \geq \end{array} \right\} \left\{ \begin{array}{l} \lambda^6 \\ \frac{1}{\lambda^6} \end{array} \right\} \times \frac{\text{div}_{d-1}}{d\kappa_d} \left( \int_{\text{bd } C} \kappa_C(x)^{(d-1)/2(d+1)} d\omega(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}} \\ &= \left\{ \begin{array}{l} \lambda^6 \\ \frac{1}{\lambda^6} \end{array} \right\} \times \frac{\text{div}_{d-1}}{d\kappa_d} \left( \int_{\text{bd } C} \kappa_C(x)^{d/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)} \frac{1}{n^{2/(d-1)}}. \end{aligned}$$

Since  $\lambda > 1$  was arbitrary, this implies the asymptotic formula (1.4).

#### 4. Proof of Theorem 2

The proofs for the cases  $\text{conv}\{x_1, \dots, x_n\}$  and  $H_C^+(x_1) \cap \dots \cap H_C^+(x_n)$  are similar; hence only the former will be considered.

**4.1** Let  $\text{bd } C$  be endowed with the geodesic metric  $\gamma_C$  and the measure  $\omega$  determined by the second fundamental form. Let  $\lambda = 2^{1/5}$ . Then there is a  $\delta > 0$  such that

$$(4.1) \quad \frac{\gamma_C(v, x)^2}{4} \leq \text{dist}(v, H_C(x)) \leq \gamma_C(v, x)^2 \text{ for } v, x \in \text{bd } C \text{ with } \gamma_C(v, x) < \delta,$$

see (3.2).

Let  $x_1, \dots \in \text{bd } C$  be a sequence with  $d_n(x_j) = d_n(\text{bd } C, \gamma_C, x_j) \rightarrow 0$  as  $n \rightarrow \infty$ . The definition of  $d_n(x_j)$  then shows that

$$(4.2) \quad \text{for all sufficiently large } n, \text{ for any } x \in \text{bd } C \text{ there is an } x_k \text{ with } k \leq n \text{ and } \gamma_C(x_k, x) < \delta \text{ and } \text{dist}(x_k, H_C(x)) = \min\{\text{dist}(x_j, H_C(x)) : j \leq n\}.$$

**4.2** By (4.2), (4.1) and the definition of  $d_n(x_j)$ ,

(4.3) for all sufficiently large  $n$  we have

$$\begin{aligned} & \delta^W(C, \text{conv}\{x_1, \dots, x_n\}) \\ &= \frac{2}{d\kappa_d} \int_{\text{bd } C} \min\{\text{dist}(x_k, H_C(x)) : k \leq n\} \kappa_C(x) d\sigma(x) \\ &\leq \frac{2}{d\kappa_d} \int_{\text{bd } C} \min\{\gamma_C(x_k, x)^2 : k \leq n\} \kappa_C(x) d\sigma(x) \\ &\leq \frac{2}{d\kappa_d} \int_{\text{bd } C} \kappa_C(x) d\sigma(x) d_n(x_j)^2 = \frac{2}{d\kappa_d} \int_{S^{d-1}} d\sigma(x) d_n(x_j)^2 \\ &= 2d_n(x_j)^2. \end{aligned}$$

Given  $n$ , let  $x_0 \in \text{bd } C$  be chosen such that  $d_n(x_j) = \min\{\gamma_C(x_0, x_k) : k \leq n\}$ . Then clearly,

$$(4.4) \quad \min\{\gamma(x_k, x) : k \leq n\} \geq d_n(x_j) - \gamma_C(x_0, x) \text{ for each } x \in D(x_0, d_n(x_j)).$$

Propositions (4.2), (4.1), (4.4), (3.1), the assumption that  $d_n(x_j) \rightarrow 0$  as  $n \rightarrow \infty$ , and Lemma 2, where  $\mu = 2$ , together imply the next statement:

(4.5) for all sufficiently large  $n$  we have the following

$$\begin{aligned}
& \delta^W(C, \text{conv}\{x_1, \dots, x_n\}) \\
&= \frac{2}{d\kappa_d} \int_{\text{bd} C} \min\{\text{dist}(x_k, H_C(x)) : k \leq n\} \kappa_C(x) d\sigma(x) \\
&\geq \frac{1}{2d\kappa_d} \int_{\text{bd} C} \min\{\gamma_C(x_k, x)^2 : k \leq n\} \kappa_C(x) d\sigma(x) \\
&\geq \frac{1}{2d\kappa_d} \int_{D(x_0, \frac{1}{2}d_n(x_j))} \left(\frac{1}{2}d_n(x_j)\right)^2 \kappa_C(x) d\sigma(x) \\
&= \frac{1}{8d\kappa_d} \int_{D(x_0, \frac{1}{2}d_n(x_j))} \kappa_C(x)^{1/2} d\omega(x) d_n(x_j)^2 \\
&\geq \frac{1}{8d\kappa_d} \min\{\kappa_C(x)^{1/2} : x \in \text{bd} C\} \frac{\kappa_{d-1}}{2} \left(\frac{d_n(x_j)}{2}\right)^{d-1} d_n(x_j)^2 \\
&= \frac{\kappa_{d-1}}{2^{d+3}d\kappa_d} \min\{\kappa_C(x)^{1/2} : x \in \text{bd} C\} d_n(x_j)^{d+1}.
\end{aligned}$$

Propositions (4.3) and (4.5) immediately give (1.9).

**4.3** Clearly, (ii) follows from Lemma 3.

**4.4** To conclude the proof of (i) we have to show that the exponents 2 and  $d + 1$  in (1.9) are best possible.

If 2 were not best possible then there were constants  $\varepsilon > 0$  and  $p > 2$  such that for the sequence  $y_1, \dots \in \text{bd} C$  from (ii),

$$\begin{aligned}
\delta^W(C, \mathcal{P}_n^i) &\leq \delta^W(C, \text{conv}\{y_1, \dots, y_n\}) \\
&\leq \varepsilon d_n(y_j)^p \leq \frac{\varepsilon \delta^p}{n^{p/(d-1)}}
\end{aligned}$$

for all sufficiently large  $n$ , in contradiction to Theorem 1.

For the proof that  $d + 1$  is best possible consider a sequence  $\varrho_1, \varrho_2, \dots \searrow 0$  and let  $x_0 \in \text{bd} C$ . Next, choose a sequence  $x_1, \dots \in \text{bd} C$  and a sequence of indices  $n_1 < n_2 < \dots$  for which the following hold:

$$(4.6) \quad d_n(x_j) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$(4.7) \quad x_1, \dots, x_{n_i} \in (\text{bd} C) \setminus D(x_0, \varrho_i) \text{ and thus } d_{n_i}(x_j) \geq \varrho_i \text{ for } i = 1, 2, \dots,$$

$$\begin{aligned}
(4.8) \quad & \int_{\text{bd} C} \min\{\gamma_C(x_k, x)^2 : k \leq n_i\} \kappa_C(x) d\sigma(x) \\
& \leq 2 \int_{D(x_0, \varrho_i)} \min\{\gamma_C(x_k, x)^2 : k \leq n_i\} \kappa_C(x) d\sigma(x).
\end{aligned}$$

Then (4.6), (4.1), (4.8), (4.7), the definition of  $d_{n_i}(x_j)$ , (3.1), (4.6) and Lemma 2 together show that



for all sufficiently large  $n_i$  the following hold:

$$\begin{aligned}
& \delta^W(C, \text{conv}\{x_1, \dots, x_{n_i}\}) \\
&= \frac{2}{d\kappa_d} \int \min\{\text{dist}(x_k, H_C(x)) : k \leq n_i\} \kappa_C(x) d\sigma(x) \\
&\leq \frac{2}{d\kappa_d} \int_{\text{bd} C} \min\{\gamma_C(x_k, x)^2 : k \leq n_i\} \kappa_C(x) d\sigma(x) \\
&\leq \frac{4}{d\kappa_d} \int \min\{\gamma_C(x_k, x)^2 : k \leq n_i\} \kappa_C(x) d\sigma(x) \\
&\leq \frac{4}{d\kappa_d} \int_{D(x_0, \varrho_i)} \kappa_C(x)^{1/2} d\omega(x) d_{n_i}(x)^2 \\
&\leq \frac{4}{d\kappa_d} \max\{\kappa_C(x)^{1/2} : x \in \text{bd} C\} \kappa_{d-1} (2d_{n_i}(x_j))^{d-1} d_{n_i}(x_j)^2 \\
&= \frac{2^{d+1} \kappa_{d-1}}{d\kappa_d} \max\{\kappa_C(x)^{1/2} : x \in \text{bd} C\} d_{n_i}(x_j)^{d+1}.
\end{aligned}$$

### 5. Proof of Theorem 3

We consider only the case of inscribed best approximating polytopes  $P_n \in \mathcal{P}_n^i$ .

**5.1** Let  $\text{bd} C$  be endowed with the geodesic metric  $\gamma_C$  and the measure  $\omega$  determined by the second fundamental form.

Let a Jordan measurable set  $J \subset \text{bd} C$  be given. Define

$$a = \left( \int_J \kappa_C(x)^{d/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)}, \quad b = \left( \int_{(\text{bd} C) \setminus J} \kappa_C(x)^{d/(d+1)} d\sigma(x) \right)^{(d+1)/(d-1)}.$$

For  $n = d + 1, \dots$ , let

$$k_n = \#(J \cap \text{vert} P_n).$$

We have to show that

$$(5.1) \quad \frac{k_n}{n} \rightarrow s_J = \frac{a^{(d-1)/(d+1)}}{a^{(d-1)/(d+1)} + b^{(d-1)/(d+1)}} \text{ as } n \rightarrow \infty.$$

**5.2** Suppose that (5.1) does not hold. Then there is a subsequence  $n_1 < n_2 < \dots$  of  $1, 2, \dots$ , such that

$$(5.2) \quad \frac{k_{n_i}}{n_i} \rightarrow t_J, \text{ say, as } i \rightarrow \infty, \text{ where } s_J \neq t_J.$$

An elementary calculation shows that

$$\begin{aligned}
(5.3) \quad & \min_{s \in [0,1]} \frac{a}{s^{2/(d-1)}} + \frac{b}{(1-s)^{2/(d-1)}} = \frac{a}{s_J^{2/(d-1)}} + \frac{b}{(1-s_J)^{2/(d-1)}} \\
& = (a^{(d-1)/(d+1)} + b^{(d-1)/(d+1)})^{(d+1)/(d-1)} = \left( \int_{\text{bd } C} \kappa_C^{d/(d+1)} d\sigma \right)^{(d+1)/(d-1)} \\
& < \frac{a}{t_J^{2/(d-1)}} + \frac{b}{(1-t_J)^{2/(d-1)}}.
\end{aligned}$$

Thus we may choose compact Jordan measurable sets  $A \subset \text{int}J$  and  $B \subset \text{int}(\text{bd } C \setminus J)$  such that for

$$a' = \left( \int_A \kappa_C^{d/(d+1)} d\sigma \right)^{(d+1)/(d-1)}, \quad b' = \left( \int_B \kappa_C^{d/(d+1)} d\sigma \right)^{(d+1)/(d-1)}$$

we still have,

$$(5.4) \quad \frac{a}{s_J^{2/(d-1)}} + \frac{b}{(1-s_J)^{2/(d-1)}} < \frac{a'}{t_J^{2/(d-1)}} + \frac{b'}{(1-t_J)^{2/(d-1)}}.$$

(int stands for interior with respect to  $\text{bd } C$ .)

The sets  $\text{vert}P_n$  are arbitrarily densely distributed on  $\text{bd } C$  if  $n$  is sufficiently large. Since  $A \subset \text{int}J$ ,  $B \subset \text{int}(\text{bd } C \setminus J)$  and  $A, B$  are compact, this shows that

$$\begin{aligned}
(5.5) \quad & \text{for all sufficiently large } n \\
& \min\{\gamma_C(v, x)^2 : v \in \text{vert}P_n\} = \min\{\gamma_C(v, x)^2 : v \in J \cap \text{vert}P_n\} \text{ for all } x \in A, \\
& \min\{\gamma_C(v, x)^2 : v \in \text{vert}P_n\} = \min\{\gamma_C(v, x)^2 : v \in (\text{bd } C \setminus J) \cap \text{vert}P_n\} \text{ for all } x \in B.
\end{aligned}$$

Now, combining the fact that in 3.3  $\lambda > 1$  was arbitrary, and propositions (3.5), (5.5), Lemma 1, (5.2), (5.4), and (5.3), we obtain the following:

$$\begin{aligned}
& \left( \int_{\text{bd } C} \kappa_C^{d/(d+1)} d\sigma \right)^{(d+1)/(d-1)} = \lim_{n \rightarrow \infty} \left\{ \left( \frac{d\kappa_d}{\text{div}_{d-1}} \delta^W(C, P_n) \right) n^{2/(d-1)} \right\} \\
& = \lim_{n \rightarrow \infty} \left\{ \left( \frac{1}{\text{div}_{d-1}} \int_{\text{bd } C} \min\{\gamma_C(v, x)^2 : v \in \text{vert}P_n\} \kappa_C d\sigma \right) n^{2/(d-1)} \right\} \\
& \geq \liminf_{n \rightarrow \infty} \left\{ \left( \frac{1}{\text{div}_{d-1}} \left( \int_A \min\{\gamma_C(v, x)^2 : v \in J \cap \text{vert}P_n\} \kappa_C d\sigma \right. \right. \right. \\
& \quad \left. \left. \left. + \int_B \min\{\gamma_C(v, x)^2 : v \in (\text{bd } C \setminus J) \cap \text{vert}P_n\} \kappa_C d\sigma \right) \right) n^{2/(d-1)} \right\} \\
& \geq \liminf_{i \rightarrow \infty} \left\{ \left( \frac{1}{\text{div}_{d-1}} \left( \inf_A \left\{ \int \min\{\gamma_C(v, x)^2 : v \in D_A\} \kappa_C d\sigma : D_A \subset \text{bd } C, \#D_A \leq k_{n_i} \right\} \right. \right. \right. \\
& \quad \left. \left. \left. + \inf_B \left\{ \int \min\{\gamma_C(v, x)^2 : v \in D_B\} \kappa_C d\sigma : D_B \subset \text{bd } C, \#D_B \leq n_i - k_{n_i} \right\} \right) \right) n_i^{2/(d-1)} \right\} \\
& \geq \left\{ \left( \int_A \kappa_C^{d/(d+1)} d\sigma \right)^{(d+1)/(d-1)} \frac{1}{t_J^{2/(d-1)}} + \left( \int_B \kappa_C^{d/(d+1)} d\sigma \right)^{(d+1)/(d-1)} \frac{1}{(1-t_J)^{2/(d-1)}} \right\}^{(d-1)/(d+1)} \\
& = \frac{a'}{t_J^{2/(d-1)}} + \frac{b'}{(1-t_J)^{2/(d-1)}} > \frac{a}{s_J^{2/(d-1)}} + \frac{b}{(1-s_J)^{2/(d-1)}} = \left( \int_{\text{bd } C} \kappa_C^{d/(d+1)} d\sigma \right)^{(d+1)/(d-1)}.
\end{aligned}$$

Contradiction.

This concludes the proof of (5.1) and the proof of Theorem 3 is complete.

## Acknowledgement

For discussions we are obliged to Professor Schneider. Many thanks are due to Professor Schnitzer for his hints.

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Stefan Glasauer  
Fachhochschule Augsburg  
Baumgartnerstr. 16  
86161 Augsburg, Germany  
stefan.glasauer(ät)fh-augsburg.de

Peter M.Gruber  
Abteilung für Analysis  
Techn. Univ. Wien  
Wiedner Hauptstraße 8 – 10/1142  
A-1040 Vienna, Austria  
pmgruber(ät)pop.tuwien.ac.at