Asymptotic approximation of smooth convex bodies by polytopes [1]

Stefan Glasauer and Rolf Schneider

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Abstract

We study asymptotic properties of the approximation of a sufficiently smooth convex body K in $\mathbb{R}^d$ by the convex hulls of n points in the boundary of K, for $n \to \infty$. The deviation is measured by the Hausdorff distance. The asymptotic distribution of the vertices of best-approximating polytopes is determined. Further results involve prescribed densities for the vertices and describe the strength of approximation by either deterministic or random polytopes.

1 Introduction and statement of results

In a well-known paper of 1975, McClure and Vitale [11] studied the approximation of sufficiently smooth convex curves in the plane by inscribed or circumscribed polygons with n vertices, for $n \to \infty$. With respect to different measures of deviation, they obtained sharp estimates of the order of convergence of best approximations, asymptotic characterizations of best approximations, and methods for the construction of asymptotically efficient approximations. It is a challenge to extend their results to higher dimensions, where different methods are required. A few of the results of [11] have been generalized in this way. We refer to Gruber [7] for a recent survey over approximation problems for convex bodies. In the present paper, we continue the line of research begun by McClure and Vitale [11] and extend two more of their theorems to sufficiently smooth convex bodies in d-dimensional space. Also a related result from [16] on random approximation will be generalized in a similar way. We restrict our considerations to approximation by inscribed polytopes and to measuring the deviation in terms of the Hausdorff metric; some hints to further possibilities will be given in Section 5.

By $\mathcal{K}^d$ we denote the set of convex bodies (non-empty, compact, convex subsets) in d-dimensional Euclidean space $\mathbb{R}^d$ ($d \geq 2$). Here $\mathbb{R}^d$ is equipped with the standard scalar product and induced norm, and $\mathcal{K}^d$ carries the Hausdorff metric, denoted by δ. (For notions and results from the theory of convex bodies that are used without explanation, we refer to [17].) A convex body $K \in \mathcal{K}^d$ is said to be of class C^k_+, where $k \geq 2$, if K has interior points and its boundary, denoted by $\operatorname{bd} K$, is a regular hypersurface of differentiability class C^k with everywhere positive curvatures. The Gauss-Kronecker curvature of K at $x \in \operatorname{bd} K$ is denoted by $\kappa_K(x)$. For $K \in \mathcal{K}^d$ and

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$n \in \mathbb{N}$, we denote by $\mathcal{P}_n(K)$ the set of convex polytopes contained in K and having at most n vertices. There exists a polytope P_n^*, not unique in general, for which

$$\delta(K, P_n^*) = \inf\{\delta(K, P) : P \in \mathcal{P}_n(K)\}.$$

We assume, without loss of generality, that P_n^* has all its vertices on the boundary of K. In the following, the constants κ_k and ϑ_k are, respectively, the volume of the k-dimensional unit ball and the minimum density of coverings of $\mathbb{R}^k$ by unit balls. By σ we denote the Euclidean surface area measure on $\operatorname{bd} K$.

We first recall an older result.

Theorem 1. *If $K \in \mathcal{K}^d$ is a convex body of class C_+^2, then*

$$\delta(K, P_n^*) \sim \frac{1}{2}\left(\frac{\vartheta_{d-1}}{\kappa_{d-1}} \int\limits_{\operatorname{bd} K} \kappa_K^{1/2}\, d\sigma\right)^{2/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{1}$$

for $n \to \infty$.

Theorem 1 was proved in [13] for bodies of class C_+^3 and was obtained by Gruber [5] under the C_+^2 assumption. It extends a result of L. Fejes Tóth [3] and Theorem 5(i) of McClure and Vitale [11] from two to d dimensions.

Theorem 1 describes the asymptotic order of approximation by the best-approximating polytopes P_n^*. While there is little hope to characterize the polytopes P_n^* individually, the asymptotic distribution of their vertices can be described precisely. This is done in the next theorem, which extends Theorem 6(i) of McClure and Vitale [11]. For its formulation, it is convenient to use some terminology from the theory of uniform distribution of sequences. In particular, we combine the notions of uniform distribution of double sequences (see Hlawka [8], p. 57) and of uniform distribution with respect to a finite measure (Kuipers and Niederreiter [10], Chapter 3) in a suitable way. Let $(S_n)_{n\in\mathbb{N}}$ be a sequence of finite subsets of $\operatorname{bd} K$ with $\operatorname{card} S_n \to \infty$ for $n \to \infty$, and let $h : \operatorname{bd} K \to \mathbb{R}^+$ be a positive continuous function on $\operatorname{bd} K$. The sequence $(S_n)_{n\in\mathbb{N}}$ is called *uniformly distributed with density* h if

$$\lim_{n\to\infty} \frac{\operatorname{card}(A \cap S_n)}{\operatorname{card} S_n} = \frac{\int_A h\, d\sigma}{\int_{\operatorname{bd} K} h\, d\sigma}$$

for all Borel sets $A \subset \operatorname{bd} K$ with $\sigma(\partial A) = 0$, where ∂ denotes the boundary relative to $\operatorname{bd} K$. By $\operatorname{vert} P$ we denote the set of vertices of the polytope P.

Theorem 2. *Let $K \in \mathcal{K}^d$ be a convex body of class C_+^2. Then the sequence $(\operatorname{vert} P_n^*)_{n\in\mathbb{N}}$ is uniformly distributed with density $\sqrt{\kappa_K}$.*

McClure and Vitale [11] also described methods for constructing asymptotically best-approximating sequences of inscribed polygons. The idea of their so-called "density approach" is to select n points on the convex curve $\operatorname{bd} K$ in such a way that they are "equally spaced" with respect to a given density, and to see afterwards which density gives the best result. More precisely, let $K \in \mathcal{K}^2$ be of class C^2_+, and let $h : \mathrm{bd}\, K \to \mathbb{R}^+$ be a continuous function such that $\int_{\mathrm{bd}\, K} h \, d\sigma = 1$. For $n \in \mathbb{N}$, choose points $x_1, \ldots, x_n$ cyclically in $\mathrm{bd}\, K$ ($x_{n+1} := x_1$) such that the arc A_j with endpoints x_j and x_{j+1} satisfies

$$\int_{A_j} h \, d\sigma = \frac{1}{n}$$

for $j = 1, \ldots, n$. Let $Q_n := \mathrm{conv}\, \{x_1, \ldots, x_n\}$. Then Theorem 7(i) of McClure and Vitale [11] implies that

$$\delta(K, Q_n) \sim \frac{1}{8} \max_{x \in \mathrm{bd}\, K} \frac{\kappa_K(x)}{h(x)^2} \frac{1}{n^2}$$

for $n \to \infty$. The choice $h(x) = \sqrt{\kappa_K(x)} / \int_{\mathrm{bd}\, K} \sqrt{\kappa_K} \, d\sigma$ yields a sequence of asymptotically best-approximating polygons.

In higher dimensions, the meaning of "equally spaced" is not so clear, and we cannot hope for a comparatively simple procedure to obtain the vertices of a sequence of asymptotically well approximating polytopes. Nevertheless, the methods used in [13] and [5] lead to an exact analogue of the result of McClure and Vitale in $\mathbb{R}^d$, although of a less constructive nature. Let q_x be the (positive definite) second fundamental form of the hypersurface $\mathrm{bd}\, K$ at the point $x \in \mathrm{bd}\, K$. We endow $\mathrm{bd}\, K$ with the Riemannian metric defined by q_x and denote by $B(x, r)$ the closed geodesic ball in $\mathrm{bd}\, K$ with centre x and radius $r > 0$. A covering $\{B(x_1, r), \ldots, B(x_n, r)\}$ of $\mathrm{bd}\, K$ by n geodesic balls of equal radii is called *minimal* if there is no such covering with balls of smaller radius. If we choose a minimal covering for each n (they clearly exist) and let Q_n be the convex hull of the centres of the corresponding balls, then $(Q_n)_{n \in \mathbb{N}}$ is a best-approximating sequence, that is

$$\delta(K, Q_n) \sim \delta(K, P_n^*) \tag{2}$$

for $n \to \infty$. This was proved in [13] (for bodies of class C^3_+) and [5]. Suppose now that a density is prescribed on $\mathrm{bd}\, K$. If we work with Riemannian metrics suitably conformal to the one above, we obtain a sequence of finite sets in $\mathrm{bd}\, K$ that is uniformly distributed with the given density, and we can give sharp estimates for the strength of approximation by the generated polytopes.

Let $K \in \mathcal{K}^d$ be of class C^2_+, and let $h : \mathrm{bd}\, K \to \mathbb{R}^+$ be a continuous function with $\int_{\mathrm{bd}\, K} h \, d\sigma = 1$. Define

$$f(x) := \left(\frac{h(x)^2}{\kappa_K(x)} \right)^{1/(d-1)} \tag{3}$$

and let R_f be the Riemannian metric on $\mathrm{bd}\, K$ which at $x \in \mathrm{bd}\, K$ is given by the quadratic form $f(x) q_x$.

Theorem 3. *For $n \in \mathbb{N}$, let $\{B_f(x_1, r), \ldots, B_f(x_n, r)\}$ be a minimal covering of $\mathrm{bd}\, K$ by n R_f-geodesic balls, and let $Q_n := \mathrm{conv}\, \{x_1, \ldots, x_n\}$. Then $(\mathrm{vert}\, Q_n)_{n \in \mathbb{N}}$ is uniformly distributed with density h, and*

$$\delta(K, Q_n) \sim \frac{1}{2} \left(\frac{\vartheta_{d-1}}{\kappa_{d-1}} \max_{x \in \mathrm{bd}\, K} \frac{\sqrt{\kappa_K(x)}}{h(x)} \right)^{2/(d-1)} \frac{1}{n^{2/(d-1)}} \tag{4}$$

for $n \to \infty$.

Remark 1. If $d = 2$ and $\{B_f(x_1, r), \ldots, B_f(x_n, r)\}$ is a minimal covering of $\operatorname{bd} K$ by n R_f-geodesic balls, then these balls do not overlap and each of them has the same R_f-volume. The volume measure induced by R_f has density $\sqrt{f\kappa_K} = h$ with respect to Euclidean arc length measure, hence the arc A_j with endpoints x_j and x_{j+1} satisfies $\int_{A_j} h\, d\sigma = 1/n$ for $j = 1, \ldots, n$. Therefore, our construction can be viewed as an extension of the "density approach" followed by McClure and Vitale.

Remark 2. The coefficient on the right-hand side of (4) attains its smallest value if and only if we choose $h(x) = \sqrt{\kappa_K(x)} / \int_{\operatorname{bd} K} \sqrt{\kappa_K}\, d\sigma$. In this case, we obtain (2) again.

Our final result concerns approximation by inscribed random polytopes. We assume that there is given a probability distribution μ on $\operatorname{bd} K$ which has a continuous density $h : \operatorname{bd} K \to \mathbb{R}^+$ with respect to the surface area measure σ. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random points on $\operatorname{bd} K$ with distribution μ. For $d = 2$, it was proved by Schneider [16] that the random polygons $P_n := \operatorname{conv}\{X_1, \ldots, X_n\}$ satisfy

$$\delta(K, P_n) \sim \frac{1}{8} \max_{x \in \operatorname{bd} K} \frac{\kappa_K(x)}{h(x)^2} \left(\frac{\log n}{n}\right)^2 \qquad \text{almost surely.} \tag{5}$$

In higher dimensions, we have an analogous result, but only under stronger differentiability assumptions and with almost sure convergence replaced by stochastic convergence.

Theorem 4. *Let* $K \in \mathcal{K}^d$ *be a convex body of class* C^3_+ *and let* $h : \operatorname{bd} K \to \mathbb{R}^+$ *be of class* C^1 *and satisfying* $\int_{\operatorname{bd} K} h\, d\sigma = 1$. *Let* $(X_n)_{n\in\mathbb{N}}$ *be a sequence of independent random points in* $\operatorname{bd} K$, *each with probability density* h *with respect to the surface area measure* σ. *If* $P_n := \operatorname{conv}\{X_1, \ldots, X_n\}$, *then*

$$P - \lim_{n\to\infty} \left(\frac{n}{\log n}\right)^{2/(d-1)} \delta(K, P_n) = \frac{1}{2} \left(\frac{1}{\kappa_{d-1}} \max_{x\in \operatorname{bd} K} \frac{\sqrt{\kappa_K(x)}}{h(x)}\right)^{2/(d-1)},$$

where $P - \lim$ *denotes stochastic convergence.*

2 Proof of Theorem 2

For the proof of Theorem 2, we need a local version of a part of Theorem 1. We assume that a convex body K of class C^2_+ and a sequence $(P^*_n)_{n\in\mathbb{N}}$ of best-approximating polytopes are given. Define

$$\varphi(A) := \int_A \kappa_K^{1/2}\, d\sigma$$

for all Borel sets $A \subset \operatorname{bd} K$, and

$$c_d := \frac{1}{2^{(d-1)/2}} \frac{\vartheta_{d-1}}{\kappa_{d-1}}.$$

A subset $A \subset \operatorname{bd} K$ will be called *Jordan measurable* if it is a Borel set and $\sigma(\partial A) = 0$.

Lemma 1. *If $A \subset \operatorname{bd} K$ is a Jordan measurable set, then*

$$\liminf_{n\to\infty} \operatorname{card}(A \cap \operatorname{vert} P_n^*)\delta(K, P_n^*)^{(d-1)/2} \ge c_d \varphi(A). \tag{6}$$

Proof. As in [13] and [5], we endow $\operatorname{bd} K$ with the Riemannian metric defined by the second fundamental form. In the following, geodesic balls in $\operatorname{bd} K$ are understood with respect to the corresponding geodesic distance. The induced Riemannian volume measure is equal to φ.

Let $k \in \mathbb{N}$. Since A is Jordan measurable, its boundary ∂A can be covered by finitely many open geodesic balls $B_1, \dots, B_{p(k)}$ with total Riemannian volume at most $1/k$. The sets

$$A_k := A \backslash \bigcup_{i=1}^{p(k)} B_i, \quad k \in \mathbb{N},$$

are Jordan measurable and satisfy $\lim_{k\to\infty} \varphi(A_k) = \varphi(A)$.

We write

$$\operatorname{card}(A \cap \operatorname{vert} P_n^*) =: m_n, \qquad \delta(K, P_n^*) =: \delta_n.$$

Proposition. Let $\lambda > 1$ be given. For all sufficiently small $\rho > 0$, the following holds. If $\delta_n \le \rho^2/2$, then the closed geodesic balls of radius $\rho\lambda^{5/2}$ and centres in $A \cap \operatorname{vert} P_n^*$ cover A_k.

This is a modification of assertion (5.3)(i) in Gruber [5] and can be proved as indicated there. Instead of the whole boundary of K, one has to consider the Jordan measurable set A_k. Moreover, the following has to be taken into account. For $x \in A$, let u_x be the outer unit normal vector of K at x and let $H^-(K, u_x) \supset K$ be the supporting halfspace of K at x. Since A_k is a compact subset of the interior of A and since $\operatorname{bd} K$ has positive curvature, there exists a number $\tau > 0$ such that

$$\operatorname{bd} K \backslash [H^-(K, u_x) - \tau u_x] \subset A \qquad \text{for all } x \in A_k.$$

Hence, for all sufficiently large n, the following holds: if $x \in A_k$ and

$$y \in \operatorname{vert} P_n^* \backslash \operatorname{int} [H^-(K, u_x) - \delta_n u_x],$$

then $y \in A$. Apart from these modifications, all the arguments of Gruber [5] to prove (5.3)(i) go through without change and result in a proof of the proposition.

For $\rho > 0$ let $n(\rho)$ be the minimum number of closed geodesic balls of radius ρ covering A_k. It follows from Lemma 1 of Gruber [5] that

$$\lim_{\rho\to 0} n(\rho)\rho^{d-1} = \frac{\vartheta_{d-1}}{\kappa_{d-1}} \varphi(A_k). \tag{7}$$

If we let $\rho_n = \sqrt{2\delta_n}$ and choose n sufficiently large, then the proposition shows that $n(\rho_n \lambda^{5/2}) \le m_n$. From (7) we therefore deduce that

$$\liminf_{i\to\infty} m_n \delta_n^{(d-1)/2} \ge \lambda^{-5(d-1)/2} c_d \varphi(A_k).$$

Letting $\lambda \to 1$ and then $k \to \infty$, we obtain the assertion (6). ■

Now we are in a position to prove Theorem 2. Let $A \subset \mathrm{bd}\, K$ be a Jordan measurable set. We set

$$\nu_n(A) := n^{-1}\mathrm{card}(A \cap \mathrm{vert}\, P_n^*)$$

and assert that

$$\limsup_{n\to\infty} \nu_n(A) \le \varphi(A)/\varphi(\mathrm{bd}\, K). \tag{8}$$

Suppose this were false. Then there exists a subsequence $(\nu_{n_i}(A))_{i\in\mathbb{N}}$ such that

$$\lim_{i\to\infty} \nu_{n_i}(A) > \varphi(A)/\varphi(\mathrm{bd}\, K). \tag{9}$$

By Theorem 1,

$$\lim_{n\to\infty} n\delta(K, P_n^*)^{(d-1)/2} = c_d\varphi(\mathrm{bd}\, K). \tag{10}$$

¿From (9) and (10) we get

$$\begin{aligned} &\lim_{i\to\infty} n_i\nu_{n_i}(A)\delta(K, P_{n_i}^*)^{(d-1)/2} \\ = &\left(\lim_{i\to\infty} \nu_{n_i}(A)\right)\left(\lim_{i\to\infty} n_i\delta(K, P_{n_i}^*)^{(d-1)/2}\right) \\ > &\ c_d\varphi(A). \end{aligned}$$

Let $A^c := \mathrm{bd}\, K\backslash A$. Since $\nu_n(A) + \nu_n(A^c) = 1$, we obtain

$$\lim_{i\to\infty} \mathrm{card}\,(A^c \cap \mathrm{vert}\, P_{n_i}^*)\delta(K, P_{n_i}^*)^{(d-1)/2} < c_d\varphi(A^c).$$

This contradicts Lemma 1. Thus (8) is proved.

¿From (8), applied to A^c, we obtain

$$\begin{aligned} &\liminf_{n\to\infty} \nu_n(A) = \liminf_{n\to\infty} [1 - \nu_n(A^c)] \\ = &\ 1 - \limsup_{n\to\infty} \nu_n(A^c) \ge 1 - \frac{\varphi(A^c)}{\varphi(\mathrm{bd}\, K)} = \frac{\varphi(A)}{\varphi(\mathrm{bd}\, K)}. \end{aligned}$$

Together with (8), this proves Theorem 2. ■

3 Proof of Theorem 3

Let $g : \mathrm{bd}\, K \to \mathbb{R}^+$ be a continuous function, and consider $\mathrm{bd}\, K$ as a Riemannian manifold, where the Riemannian metric at $x \in \mathrm{bd}\, K$ is given by $g(x)q_x$. (As in the introduction, q_x denotes the second fundamental form at x.) By $B_g(x, r)$ we denote the closed geodesic ball with centre x and radius r that corresponds to this Riemannian metric. If $g \equiv 1$, we denote this ball simply by $B(x, r)$. For $n \in \mathbb{N}$ define $\rho_g(n)$ as the smallest number $r > 0$ such that there are points $x_1, \dots, x_n \in \mathrm{bd}\, K$ with $\mathrm{bd}\, K = \bigcup_{i=1}^n B_g(x_i, r)$. Lemma 1 of Gruber [5] implies that

$$\lim_{n\to\infty} n\rho_g(n)^{d-1} = \frac{\vartheta_{d-1}}{\kappa_{d-1}} \int\limits_{\mathrm{bd}\, K} g^{(d-1)/2}\kappa_K^{1/2}\, d\sigma. \tag{11}$$

Another result of Gruber that we shall need in this and the next section is the following lemma.

Lemma 2. *For every $\lambda > 1$ there is a number $r_0 > 0$ such that for all $r \in (0, r_0)$ the following implications hold:*

$$\operatorname{bd} K = \bigcup_{i=1}^{n} B(x_i, r) \quad \Rightarrow \quad \delta(K, \operatorname{conv}\{x_1, \ldots, x_n\}) \le \frac{1}{2}\lambda r^2 ,$$

$$\operatorname{bd} K \ne \bigcup_{i=1}^{n} B(x_i, r) \quad \Rightarrow \quad \delta(K, \operatorname{conv}\{x_1, \ldots, x_n\}) \ge \frac{1}{2\lambda} r^2 .$$

This lemma is a reformulation of results obtained by Gruber [5] in the course of the proof of his Theorem 3.

Now let $h : \operatorname{bd} K \to \mathbb{R}^+$ be a continuous function with $\int_{\operatorname{bd} K} h \, d\sigma = 1$, define f by (3), and set

$$\alpha = \min_{x \in \operatorname{bd} K} f(x).$$

Let $\lambda > 1$ be given. Let U, V be nonempty open subsets of $\operatorname{bd} K$ such that $f(x) \le \lambda\alpha$ for $x \in U$ and $\operatorname{cl} V \subset U$. For $n \in \mathbb{N}$, choose points $x_1^n, \ldots, x_n^n \in \operatorname{bd} K$ such that $\operatorname{bd} K = \bigcup_{i=1}^n B_f(x_i^n, \rho_f(n))$. We assert that

$$V \not\subset \bigcup_{i=1}^{n} B_f(x_i^n, \rho_f(n)/\sqrt{\lambda}) \tag{12}$$

for all sufficiently large n.

For the proof, let $\emptyset \ne W \subset \operatorname{bd} K$ be an open set with $\operatorname{cl} W \subset V$. Choose a continuous function $g : \operatorname{bd} K \to \mathbb{R}^+$ with $f \le g \le \lambda f$, $g(x) > f(x)$ for all $x \in W$ and $g(x) = f(x)$ for $x \notin W$. We have $B_f(x, r/\sqrt{\lambda}) \subset B_g(x, r)$ for all $x \in \operatorname{bd} K$ and all $r > 0$. Since $\rho_f(n) \to 0$ for $n \to \infty$, for all sufficiently large $n \in \mathbb{N}$ the ball $B_f(x_i, \rho_f(n))$ is contained in V if it meets W.

Suppose now that (12) were false. Then the balls $B_g(x_i^n, \rho_f(n))$ $(i = 1, \ldots, n)$ form a covering of $\operatorname{bd} K$ for infinitely many $n \in \mathbb{N}$. For these n we have $\rho_g(n) \le \rho_f(n)$. The asymptotic relation (11) together with $f < g$ on W yields

$$\lim_{n \to \infty} n \rho_g(n)^{d-1} \le \lim_{n \to \infty} n \rho_f(n)^{d-1} < \lim_{n \to \infty} n \rho_g(n)^{d-1}.$$

This contradiction proves (12).

Since $f(x) \le \lambda\alpha$ for all $x \in U$, we have

$$B(x, r) \subset B_f(x, \sqrt{\lambda\alpha}\, r) \tag{13}$$

for all $x \in \operatorname{bd} K$ and all $r > 0$ with $B(x, r) \subset U$. For all sufficiently large $n \in \mathbb{N}$ the ball $B(x_i^n, \rho_f(n)/\lambda\sqrt{\alpha})$ is contained in U if it meets V. Therefore it follows from (12) and (13) that

$$\operatorname{bd} K \ne \bigcup_{i=1}^{n} B\left(x_i^n, \rho_f(n)/\lambda\sqrt{\alpha}\right) ,$$

if n is sufficiently large. Hence for $Q_n := \operatorname{conv}\{x_1^n, \ldots, x_n^n\}$ we see from Lemma 2 that

$$\delta(K, Q_n) \geq \frac{1}{2\alpha\lambda^3}\, \rho_f(n)^2$$

for all sufficiently large n.

On the other hand, we have $B_f(x, r) \subset B(x, r/\sqrt{\alpha})$ because of $f \geq \alpha$. Thus the balls $B(x_i^n, \rho_f(n)/\sqrt{\alpha})$ $(i = 1, \ldots, n)$ form a covering of $\operatorname{bd} K$, and for all sufficiently large n we get

$$\delta(K, Q_n) \leq \frac{\lambda}{2\alpha}\, \rho_f(n)^2 \,,$$

by Lemma 2. Since $\lambda > 1$ was arbitrary, we deduce that

$$\lim_{n\to\infty} \frac{\delta(K, Q_n)}{\rho_f(n)^2} = \frac{1}{2\alpha}\,.$$

¿From (11) and the definition (3) of f we get the assertion (4) of Theorem 3.

The proof of the fact that $(\operatorname{vert} Q_n)_{n\in\mathbb{N}}$ is uniformly distributed with density h is similar to the proof of Theorem 2, but simpler. One has to replace φ by the measure

$$\varphi_f(A) := \int_A f^{(d-1)/2} \kappa_K^{1/2}\, d\sigma = \int_A h\, d\sigma;$$

and the role of Theorem 1 and Lemma 1 in the proof of Theorem 2 is played by Gruber's relation (11). ■

4 Proof of Theorem 4

The proof of Theorem 4 relies heavily on the following result of Janson [9] (Theorem 1.2 and Remark 4).

Lemma 3. *Let M be a $(d-1)$-dimensional Riemannian manifold of class C^2 with Riemannian metric of class C^1. Let the volume measure be denoted by μ. Let $V \subset M$ be Jordan measurable and let $U \subset M$ be an open set with $\operatorname{cl} V \subset U$ such that $0 < \mu(V) < \mu(U) < \infty$. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent, identically distributed random points in U with distribution $(\mu|U)/\mu(U)$. For $r > 0$ set*

$$N_r := \min\{n \in \mathbb{N} : V \subset \bigcup_{i=1}^{n} B(X_i, r)\}\,,$$

where $B(x, r)$ is the closed geodesic ball with respect to the given metric. Then for $x \in \mathbb{R}$ we have

$$\lim_{r\to 0} P\left[\frac{\kappa_{d-1} r^{d-1}}{\mu(U)} N_r - \log \frac{\mu(V)}{\kappa_{d-1} r^{d-1}} - (d-1) \log\log \frac{\mu(V)}{\kappa_{d-1} r^{d-1}} - \log c \leq x\right] = e^{-e^{-x}},$$

where the constant c can be obtained from [9], *formula* (9.24).

As a corollary we get the following lemma.

Lemma 4. *With the above notation and with*

$$D_n := \min\{r > 0 \,:\, V \subset \bigcup_{i=1}^{n} B(X_i, r)\}$$

we have

$$P - \lim_{n\to\infty} \frac{n}{\log n}\, D_n^{d-1} = \frac{\mu(U)}{\kappa_{d-1}}\,.$$

Proof. For given $n \in \mathbb{N}$ and $x \in \mathbb{R}$ there is a unique number $a_{n,x} \in \left(0,\ \sqrt[d-1]{\mu(V)/\kappa_{d-1}}\right)$ satisfying the equation

$$n = \frac{\mu(U)}{\kappa_{d-1} a_{n,x}^{d-1}} \left(\log \frac{\mu(V)}{\kappa_{d-1} a_{n,x}^{d-1}} + (d-1) \log\log \frac{\mu(V)}{\kappa_{d-1} a_{n,x}^{d-1}} + \log c + x \right).$$

Now obviously Lemma 3 implies

$$\lim_{n\to\infty} P\left[D_n \le a_{n,x}\right] = e^{-e^{-x}}\,.$$

Let $\varepsilon > 0$ and $x \in \mathbb{R}$. Since $a_{n,x}$ is increasing in x, it is easy to see that for all $b \in \mathbb{R}^+$ there is a number $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ we have $(1+\varepsilon)a_{n,x} \ge a_{n,x+b}$. This yields $\lim_{n\to\infty} P\left[D_n \le (1+\varepsilon)a_{n,x}\right] = 1$. Analogously we get $\lim_{n\to\infty} P\left[D_n \ge (1-\varepsilon)a_{n,x}\right] = 1$. An easy computation yields

$$\lim_{n\to\infty} \frac{n}{\log n}\, a_{n,x}^{d-1} = \frac{\mu(U)}{\kappa_{d-1}},$$

and the assertion follows. ■

Proof of Theorem 4. Let K be a convex body of class C^3_+, let the density function h be of class C^1 and let the function $f : \operatorname{bd} K \to \mathbb{R}^+$ be given by (3). The volume measure μ on $\operatorname{bd} K$ with respect to the Riemannian metric $f(x)q_x$ ($x \in \operatorname{bd} K$) is the distribution of the random points X_i. The metric is of class C^1. As in the proof of Theorem 3, let $\lambda > 1$, $\alpha := \min_{x\in \operatorname{bd} K} f(x)$, let $\emptyset \ne U \subset \operatorname{bd} K$ be open so that $f(x) \le \lambda\alpha$ for $x \in U$. Let $V \subset U$ be a compact, Jordan measurable set with $\mu(V) > 0$. Define the random variable

$$M_n := \operatorname{card}\{i \in \mathbb{N} :\, i \le n,\ X_i \in U\}\,.$$

According to the strong law of large numbers we have

$$\lim_{n\to\infty} \frac{M_n}{n} = \mu(U) \qquad \text{almost surely.} \tag{14}$$

Let $i_1 \in \mathbb{N}$ be the smallest number with $X_{i_1} \in U$, and for $n \ge 1$ let $i_{n+1} \in \mathbb{N}$ be minimal with $i_{n+1} \ge i_n + 1$ and $X_{i_{n+1}} \in U$. Almost surely i_n is defined for all $n \in \mathbb{N}$, and $(X_{i_n})_{n\in\mathbb{N}}$ is a sequence of i.i.d. points in U with distribution $(\mu|U)/\mu(U)$. Define

$$\begin{aligned} D_n &:= \min\{r>0 \,:\, \operatorname{bd} K = \bigcup_{i=1}^n B_f(X_i, r)\}\,, \\ \widetilde{D}_n &:= \min\{r>0 \,:\, V \subset \bigcup_{j=1}^n B_f(X_{i_j}, r)\}\,. \end{aligned}$$

¿From Lemma 4 it follows that

$$P - \lim_{n\to\infty} \frac{n}{\log n}\, D_n^{d-1} = \frac{1}{\kappa_{d-1}}\,, \tag{15}$$

$$P - \lim_{n\to\infty} \frac{M_n}{\log M_n}\, \widetilde{D}_{M_n}^{d-1} = \frac{\mu(U)}{\kappa_{d-1}}\,. \tag{16}$$

¿From (14) und (16) we derive that

$$P - \lim_{n\to\infty} \frac{n}{\log n}\, \widetilde{D}_{M_n}^{d-1} = \frac{1}{\kappa_{d-1}},$$

and with (15) we get $\lim_{n\to\infty} P[D_n/\lambda < \widetilde{D}_{M_n}] = 1$. This yields

$$\lim_{n\to\infty} P\left[V \not\subset \bigcup_{i\le n,\, X_i\in U} B_f(X_i, D_n/\lambda)\right] = 1\,.$$

A similar consideration as in the proof of Theorem 3 gives

$$\lim_{n\to\infty} P\left[\operatorname{bd} K \ne \bigcup_{i=1}^n B\left(X_i, D_n/\sqrt{\lambda^3\alpha}\right)\right] = 1\,,$$

and an application of Lemma 2 shows

$$\lim_{n\to\infty} P\left[\delta(K, P_n) \ge \frac{D_n^2}{2\lambda^4\alpha}\right] = 1\,.$$

Analogously we get

$$\lim_{n\to\infty} P\left[\delta(K, P_n) \le \lambda\,\frac{D_n^2}{2\alpha}\right] = 1$$

and therefore

$$P - \lim_{n\to\infty} \frac{\delta(K, P_n)}{D_n^2} = \frac{1}{2\alpha}\,.$$

Because of (15) this implies the desired result. ■

5 Similar results

Approximation of convex bodies of class C^2_+ by circumscribed polytopes with n facets can be treated in a similar way. A counterpart to Theorem 1 was obtained in Schneider [15]. Combining the methods of [15], [5] and the present paper, further results of this nature can be proved. For example, if $x_1, \dots, x_n$ are as in Theorem 3, we define $Q_{(n)} := \bigcap_{i=1}^n H_i^-$, where H_i^- is the supporting halfspace of K at x_i. If n is sufficiently large, $Q_{(n)}$ is bounded, and the limit of $n^{2/(d-1)}\delta(K, Q_{(n)})$ is the same as in Theorem 3. In a similar way, one can prove counterparts to Theorems 2 and 4.

For the Banach-Mazur notion of distance, for which approximation was investigated by Gruber [5], an analogue to Theorem 2 can be obtained along similar lines.

For approximation in the sense of the distance notion introduced in Schneider [14], corresponding procedures are possible. For a convex body K and a d-dimensional polytope $P \subset K$, the distance $\delta^A(K, P)$ is the maximal volume of a cap $K \cap H^+$, where H^+ is a closed halfspace that is determined by a facet of P and does not contain the interior of P. Best approximation in the sense of δ^A was investigated in [14]; further results concerning δ^A in the plane are found in Müller [12].

We now briefly describe a construction of well-approximating polytopes with a given number of facets, with respect to this distance δ^A and a prescribed density. Let K be a convex body and h a function, satisfying the assumptions of Theorem 3. Define $f_A : \mathrm{bd}\, K \to \mathbb{R}^+$ by

$$f_A(x) := \left(\frac{h(x)^2}{\kappa_K(x)^{2/(d+1)}} \right)^{1/(d-1)}. \tag{17}$$

Let r_x be the first fundamental form of equiaffine differential geometry of $\mathrm{bd}\, K$ at the point $x \in \mathrm{bd}\, K$ (see, e.g., Blaschke [2] for a definition) and endow $\mathrm{bd}\, K$ with the Riemannian metric $f_A(x) r_x$. The corresponding volume measure has density h with respect to the (Euclidean) surface area measure. Let now $B_A(x, r)$ be the closed geodesic ball with respect to this metric ($x \in \mathrm{bd}\, K$, $r > 0$), and let $\rho_A : \mathbb{N} \to \mathbb{R}^+$ be defined analogously as in the proof of Theorem 3. For $n \in \mathbb{N}$ consider points $x_1^n, \dots, x_n^n \in \mathrm{bd}\, K$ with $\mathrm{bd}\, K = \bigcup_{i=1}^n B_A(x_i^n, \rho_A(n))$. Let H_i^- and u_i be the supporting halfspace and the outer unit normal vector of K at x_i^n, respectively. Let η_i be the smallest positive number such that $B_A(x_i, \rho_A(n)) \cap \mathrm{int}\,(H_i^- - \eta_i u_i) = \emptyset$, and set

$$\widetilde{P}_n := \bigcap_{i=1}^n (H_i^- - \eta_i u_i)\,.$$

If n is sufficiently large, $\widetilde{P}_n$ is a d-dimensional polytope with at most n facets, which is contained in K since the geodesic balls cover $\mathrm{bd}\, K$. In analogy to equation (4) of Theorem 3, one then obtains the asymptotic relation

$$\lim_{n\to\infty} n^{(d+1)/(d-1)}\, \delta^A(K, \widetilde{P}_n) = \frac{\kappa_{d-1}}{d+1} \left(\frac{\vartheta_{d-1}}{\kappa_{d-1}} \max_{x \in \mathrm{bd}\, K} \frac{\kappa_K(x)^{1/(d+1)}}{h(x)} \right)^{(d+1)/(d-1)}.$$

In order to state a result corresponding to Theorem 4 for the distance notion δ^A, we consider random polytopes contained in K constructed in the following way. Let

$K \subset \mathbb{R}^d$ ($d \geq 2$) and h be a convex body and a function, respectively, as in Theorem 4. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent random points on $\mathrm{bd}\, K$, distributed as in Theorem 4. Now consider a covering $B_A(X_i, r)$ ($i = 1, \ldots, n$) of $\mathrm{bd}\, K$ by n geodesic balls with respect to the Riemannian metric on $\mathrm{bd}\, K$ introduced above, and let $r > 0$ be as small as possible. Set

$$\widetilde{Q}_n := \bigcap_{i=1}^{n} (H_i^- - \eta_i u_i),$$

where the notaton is the same as above and $\eta_i > 0$ is chosen minimal with $B_A(X_i, r) \cap \mathrm{int}\,(H_i^- - \eta_i u_i) = \emptyset$. It is easy to see that almost surely $\widetilde{Q}_n \subset K$ is a d-dimensional polytope for all sufficiently large $n \in \mathbb{N}$. The following asymptotic relation can be proved:

$$P - \lim_{n\to\infty} \left(\frac{n}{\log n}\right)^{(d+1)/(d-1)} \delta^A(K, \widetilde{Q}_n) = \frac{\kappa_{d-1}}{d+1} \left(\frac{1}{\kappa_{d-1}} \max_{x\in \mathrm{bd}\, K} \frac{\kappa_K(x)^{1/(d+1)}}{h(x)}\right)^{(d+1)/(d-1)}.$$

Of the other measures of deviation studied by McClure and Vitale [11] for inscribed and circumscribed polygons, the area or perimeter difference are of particular interest. In higher dimensions, these functionals can naturally be replaced by volume, surface area, mean width, or any of the other quermassintegrals. Precise asymptotic estimates for the approximation of sufficiently smooth convex bodies are, in general, hard to obtain. For the volume difference of inscribed or circumscribed polytopes, such results have recently been proved by Gruber [6]; for the mean width, see [4]. The case of the surface area seems to be difficult. Concerning the asymptotic approximation by convex hulls of random points, some more references are given in the survey [16]. For convex hulls of independent uniform random points in the interior of a convex body of class C^3_+, the asymptotic behaviour of all the quermassintegrals was determined by Bárány [1].

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Author's addresses:

Stefan Glasauer
Fachhochschule Augsburg
Baumgartnerstr. 16
86161 Augsburg, Germany
stefan.glasauer(ät)fh-augsburg.de

Rolf Schneider
Mathematisches Institut
Albert-Ludwigs-Universität
Eckerstr. 1
79104 Freiburg i. Br., Germany