Translative and kinematic integral formulae concerning the convex hull operation

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Abstract. For convex bodies K, K' and a translation τ in n-dimensional Euclidean space, let $K \vee \tau K'$ be the convex hull of the union of K and $\tau K'$. Let F be a geometric functional on the space of all convex bodies. We consider special families $(\alpha_r)_{r>0}$ of measures on the translation group T_n such that the limit

$$\lim_{r\to\infty} \int_{T_n} F(K \vee \tau K')\, d\alpha_r(\tau)$$

exists and can be expressed in terms of K and K'. The functionals F under consideration are derived from the mixed volume or the mixed area measure functional. Analogous questions are treated for the motion group instead of the translation group. The resulting relations can be regarded as dual counterparts to various versions of the principal kinematic formula. Motivation for our investigations is provided by classical and recent results from spherical integral geometry.

Key words: Convex body, convex hull operation, mixed area measure, mixed volume, generalized curvature measure, boundary structure, principal kinematic formula, spherical integral geometry.

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The classical results of convexity-related integral geometry concern mean value formulae for intersections, orthogonal projections and Minkowski sums of convex bodies. One further basic operation, which is in some sense dual to the intersection, is defined by the convex hull of the union of two convex bodies. Integral formulae concerning this operation are known in spherical integral geometry, as intersection formulae can be transferred, due to the spherical principle of duality, into results regarding the convex hull operation. It is clear that, if there are corresponding formulae in Euclidean space, they must be of a different nature because of the non-compactness of Euclidean space and its motion group. It is shown in this article that there are kinematic formulae, in the form of limit relations, which resemble these spherical results. They have particularly nice properties with respect to Minkowski addition, and they are related to the well-known rotation sum and projection formulae. In a first step, we prove translative versions, which have a simpler structure than the known translative intersection formulae. It is interesting that the invariant measure on the translation group (Lebesgue measure) can be replaced by much more general measures while still leading to simple explicit results. We then apply known rotational mean value formulae to deduce kinematic versions from our translative results.

For integral geometry of spherically convex bodies, we refer to the thesis [3], which contains classical as well as new results; see also Part IV of Santaló's book [6] and the literature quoted there. For integral geometry of convex bodies in Euclidean space, we recommend the recent survey by Schneider & Wieacker [10], or Schneider & Weil [9].

In Section 1 we recall a recent result from spherical integral geometry, which provided motivation for the present work. Our results are stated in Section 2. Sections 3 and 4 contain the proofs.

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1 A recent result from spherical integral geometry

We want to state two kinematic formulae for the support measures of spherically convex bodies. The relevant notions can be introduced as follows (see [3]).

Let S^n be n-dimensional spherical space, i.e. the set of all unit vectors in $\mathbb{R}^{n+1}$, equipped with the standard inner product $\langle\cdot,\cdot\rangle$. For $x,y\in S^n$ we let $[x,y]^s := \mathrm{pos}\{x,y\}\cap S^n$, where pos is the positive hull operation. A subset $K\subset S^n$ is, by definition, in the class $\mathcal{K}^s$ of all spherically convex bodies, if it is compact and if $[x,y]^s\subset K$ for all $x,y\in K$. If $K\in\mathcal{K}^s$, then also $K^* := \{x\in S^n : \langle x,y\rangle \le 0 \text{ for all } y\in K\}\in\mathcal{K}^s$; this body is called the polar body of K. For $K,K'\in\mathcal{K}^s$, the body $(K^*\cap K'^*)^*\in\mathcal{K}^s$ equals the union of all sets $[x,y]^s$ for $x\in K$, $y\in K'$; we denote this body by $K\vee K'$. We let $\mathrm{Nor}K := \{(x,u)\in K\times K^* : \langle x,u\rangle = 0\}$ and call the elements of $\mathrm{Nor}K$ the support elements of K. For $K\in\mathcal{K}^s$ and $j\in\{0,\dots,n-1\}$, the support measures $\Theta_j^s(K,\cdot)$ can be defined as the uniquely determined Borel measures on $S^n\times S^n$ that are concentrated on $\mathrm{Nor}K$ and satisfy the Steiner-type relation

$$\int\limits_{S^n\setminus(K\cup K^*)} f\,d\sigma = \sum_{j=0}^{n-1}\beta_j\beta_{n-j-1}\int\limits_0^{\pi/2}\cos^j t\sin^{n-j-1} t\int\limits_{S^n\times S^n} f(x\cos t+u\sin t)\,d\Theta_j^s(K,(x,u))dt$$

for all σ-integrable functions $f: S^n\to\mathbb{R}$. Here σ is spherical Lebesgue measure on S^n and $\beta_n := \sigma(S^n)$.

We need some further notations. By ν we denote the normalized invariant measure on the group SO_{n+1} of all proper rotations in $\mathbb{R}^{n+1}$, and for $\vartheta\in SO_{n+1}$ and subsets $\eta,\eta'\subset S^n\times S^n$ we let

$$\begin{aligned}
\eta\wedge\vartheta\eta' &:= \big\{(x,u)\in S^n\times S^n &:\ & u\in[u_1,u_2]^s \text{ with } u_1,u_2\in S^n,\ (x,u_1)\in\eta,\\
&&& (\vartheta^{-1}x,\vartheta^{-1}u_2)\in\eta'\big\},\\
\eta\vee\vartheta\eta' &:= \big\{(x,u)\in S^n\times S^n &:\ & x\in[x_1,x_2]^s \text{ with } x_1,x_2\in S^n,\ (x_1,u)\in\eta,\\
&&& (\vartheta^{-1}x_2,\vartheta^{-1}u)\in\eta'\big\}.
\end{aligned}$$

The following result was proved in [3]; note that we consider sums over empty index sets to be zero. We assume the measure $\Theta_j^s(K,\cdot)$ to be complete, i.e. its domain is extended to include the sets $\eta_1\cup\eta_2$, where η_1 is a Borel set and η_2 is a subset of a Borel set of $\Theta_j^s(K,\cdot)$-measure zero.

Theorem 1. *Let $K,K'\in\mathcal{K}^s$ and let $\eta\subset\mathrm{Nor}K$ and $\eta'\subset\mathrm{Nor}K'$ be Borel sets. Let $j\in\{0,\dots,n-1\}$. Then we have*

$$\int\limits_{SO_{n+1}}\Theta_j^s(K\cap\vartheta K',\eta\wedge\vartheta\eta')\,d\nu(\vartheta) = \sum_{i=j+1}^{n-1}\Theta_i^s(K,\eta)\Theta_{n+j-i}^s(K',\eta'), \tag{1}$$

$$\int\limits_{SO_{n+1}}\Theta_j^s(K\vee\vartheta K',\eta\vee\vartheta\eta')\,d\nu(\vartheta) = \sum_{i=0}^{j-1}\Theta_i^s(K,\eta)\Theta_{j-i-1}^s(K',\eta'). \tag{2}$$

Each of these two formulae follows immediately from the respective other one since the support measures satisfy $\Theta_j^s(K,\eta)=\Theta_{n-j-1}^s(K^*,\eta^{-1})$, where $\eta^{-1} := \{(u,x) : (x,u)\in\eta\}$. This theorem provided motivation for the work [4], where analogs of (1) in Euclidean space were examined, as well as for the present paper, in which we aim at establishing Euclidean counterparts for (2).

2 Statement of results

We begin by fixing our notation; Schneider's book [7] is our basic reference for the notions used below. By $\mathcal{K}$ we denote the space of all convex bodies, i.e. non-empty, compact, convex subsets of Euclidean space $\mathbb{R}^n$. For $K_1, \ldots, K_n \in \mathcal{K}$, let $V(K_1, \ldots, K_n)$ be the mixed volume and $S(K_1, \ldots, K_{n-1}, \cdot)$ the mixed area measure, which is a finite measure on the unit sphere S^{n-1}. (Unless stated otherwise, all measures will be Borel measures, i.e. they are defined on the σ-algebra $\mathcal{B}(X)$ of all Borel sets of a topological space X.) For $K \in \mathcal{K}$, the support function of K is denoted by $h_K : S^{n-1} \to \mathbb{R}$, i.e. $h_K(u) = \max\{\langle u, x\rangle : x \in K\}$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$. Mixed volumes and area measures are related by means of the equation

$$V(K_1, \ldots, K_n) = \frac{1}{n} \int_{S^{n-1}} h_{K_1}\, dS(K_2, \ldots, K_n, \cdot). \tag{3}$$

We will abbreviate, e.g., $V(K_1, \ldots, K_1, K_{j+1}, \ldots, K_n)$ by $V(K[j], K_{j+1}, \ldots, K_n)$ and use analogous conventions in the case of the mixed area measures. For $j \in \{0, \ldots, n-1\}$, the j-th area measure of a convex body K is defined as

$$S_j(K, \cdot) := S(K[j], B^n[n-j-1], \cdot),$$

where B^n is the unit ball in $\mathbb{R}^n$. For $K, K' \in \mathcal{K}$, we will write $K \vee K'$ for the convex hull of the union $K \cup K'$. The support function of the convex body $K \vee K'$ is given by

$$h_{K \vee K'}(u) = \max\{h_K(u), h_{K'}(u)\}, \qquad u \in S^{n-1}.$$

We will repeatedly use the following fact: If $h : S^{n-1} \to \mathbb{R}$ is such that its positively homogeneous extension of degree 1 to $\mathbb{R}^n$ is a convex function, then there is a unique convex body K with $h = h_K$ (see [7], Theorem 1.7.1).

Our first results are translative formulas for the mixed volumes and for the mixed area measures concerning the "convex hull operation", where the integration is with respect to quite general measures on $\mathbb{R}^n$. A measure α on $\mathbb{R}^n$ is called homogeneous of degree $\delta > 0$, if $\alpha(sA) = s^\delta \alpha(A)$ for all $s \geq 0$ and all $A \in \mathcal{B}(\mathbb{R}^n)$.

Theorem 2. *Let α be a locally finite measure on $\mathbb{R}^n$ which is homogeneous of degree $\delta > 0$. Define $Z \in \mathcal{K}$ to be the convex body whose support function is $\int_{B^n} |\langle \cdot, x\rangle|\, d\alpha(x)$. Let $j \in \{1, \ldots, n\}$ and $K_{j+1}, \ldots, K_n \in \mathcal{K}$. Let $f : S^{n-1} \to \mathbb{R}$ be continuous. Let $a > 0$. Then we have*

$$\lim_{r\to\infty} \frac{1}{r^{\delta+1}} \int_{rB^n} V(K \vee (K'+x)[j], K_{j+1}, \ldots, K_n)\, d\alpha(x)$$
$$= \frac{1}{2} \sum_{i=0}^{j-1} V(K[i], K'[j-i-1], K_{j+1}, \ldots, K_n, Z),$$

$$\lim_{r\to\infty} \frac{1}{r^{\delta+1}} \int_{rB^n} \int_{S^{n-1}} f\, dS(K \vee (K'+x)[j], K_{j+1}, \ldots, K_{n-1}, \cdot)\, d\alpha(x)$$
$$= \frac{1}{2} \sum_{i=0}^{j-1} \int_{S^{n-1}} f\, dS(K[i], K'[j-i-1], K_{j+1}, \ldots, K_{n-1}, Z, \cdot),$$

uniformly for all $K, K' \in \mathcal{K}$ which are contained in the ball aB^n.

The body Z in Theorem 2 is a zonoid, i.e. the limit (in the Hausdorff metric) of vector sums of line segments. This follows from the proof of Theorem 2 and the fact that a convex body is a zonoid if its support function has a representation $\int_{S^{n-1}} |\langle \cdot, u\rangle|\, d\rho(u)$ with a finite measure ρ on S^{n-1}, see Theorem 3.5.2 in Schneider [7].

Remark. Using the methods applied in the proof of Theorem 2, iterated versions can be obtained inductively. We state the version for the mixed volumes. Let $j \in \{1, \ldots, n\}$, $k \in \{1, \ldots, j\}$, and let $K_0, \ldots, K_k$, $K_{j+1}, \ldots, K_n \in \mathcal{K}$. Define a function $g : (\mathbb{R}^n)^k \to \mathbb{R}$ by

$$g(x_1, \ldots, x_k) := V(K_0 \vee (K_1 + x_1) \vee \cdots \vee (K_k + x_k)[j], K_{j+1}, \ldots, K_n),$$

where the operation "$\vee$" is to be associative. Let $\alpha_1, \ldots, \alpha_k$ be locally finite measures on $\mathbb{R}^n$ such that α_i is homogeneous of degree $\delta_i > 0$. Define $Z_i \in \mathcal{K}$ by $h_{Z_i} = \int_{B^n} |\langle \cdot, x \rangle| \, d\alpha_i(x)$. Then we have

$$\begin{aligned} &\lim_{r \to \infty} \frac{1}{r^{\delta_1 + \cdots + \delta_k + k}} \int_{rB^n} \cdots \int_{rB^n} g(x_1, \ldots, x_k) \, d\alpha_1(x_1) \cdots d\alpha_k(x_k) \\ &= \frac{1}{2^k} \sum_{\substack{r_0, \ldots, r_k = 0 \\ r_0 + \cdots + r_k = j-k}}^{j-k} V(K_0[r_0], \ldots, K_k[r_k], K_{j+1}, \ldots, K_n, Z_1, \ldots, Z_k), \end{aligned}$$

uniformly for all $K_0, \ldots, K_k \in \mathcal{K}$ which are contained in a fixed ball. We will indicate the proof of this formula in Section 3 after the proof of Theorem 2.

Using known rotational mean value formulae for the mixed volumes, we can deduce a kinematic version from Theorem 2. Let SO_n be the group of all proper rotations of $\mathbb{R}^n$, and denote its invariant probability measure by ν. The group G_n of all proper rigid motions consists of all maps $g_{x,\vartheta} : \mathbb{R}^n \to \mathbb{R}^n$, defined by $g_{x,\vartheta}(y) = \vartheta(y) + x$, for $x \in \mathbb{R}^n$, $\vartheta \in SO_n$. If α is a locally finite measure on $\mathbb{R}^n$, we denote by μ_α the image of the product measure $\alpha \otimes \nu$ on $\mathbb{R}^n \times SO_n$ under the map $(x, \vartheta) \mapsto g_{x,\vartheta}$. By κ_n we denote the volume of the unit ball in $\mathbb{R}^n$. We state the result for the mixed volumes only and remark that, as above, iterated versions can also be obtained.

Theorem 3. *Under the assumptions of Theorem 2, we have*

$$\begin{aligned} &\lim_{r \to \infty} \frac{1}{r^{\delta+1}} \int_{\{g \in G_n : gK' \subset rB^n\}} V(K \vee gK'[j], K_{j+1}, \ldots, K_n) \, d\mu_\alpha(g) \\ &= \frac{1}{2\kappa_n} \sum_{i=0}^{j-1} V(K[i], B^n[j-i-1], K_{j+1}, \ldots, K_n, Z) \; V(K'[j-i-1], B^n[n-j+i+1]), \end{aligned}$$

uniformly for all $K, K' \in \mathcal{K}$ contained in the ball aB^n.

Remark. One can obtain explicit formulas for even more general measures on the motion group. We consider only the case of the volume functional V_3 for convex bodies in $\mathbb{R}^3$. Let the assumptions of Theorem 2 be given. Let $\tilde{\nu}$ be a finite measure on SO_3. Denote by $\mu_{\alpha,\tilde{\nu}}$ the image of the product measure $\alpha \otimes \tilde{\nu}$ on $\mathbb{R}^3 \times SO_3$ under the map $(x, \vartheta) \mapsto g_{x,\vartheta}$. For arbitrary $K \in \mathcal{K}$, let $R_{\tilde{\nu}} K \in \mathcal{K}$ be the body with support function $\int_{SO_3} h_{\vartheta K} \, d\tilde{\nu}(\vartheta)$. We denote the image of $\tilde{\nu}$ under the map $\vartheta \mapsto \vartheta^{-1}$ by $\tilde{\nu}^{-1}$. We now deduce from Theorem 2 and Eq. (3) that

$$\begin{aligned} &\lim_{r \to \infty} \frac{2}{r^{\delta+1}} \int_{\{g \in G_3 : gK' \subset rB^3\}} V_3(K \vee gK') \, d\mu_{\alpha,\tilde{\nu}}(g) \\ &\qquad = \; V(K, K, Z) + V(K, R_{\tilde{\nu}} K', Z) + V(K', K', R_{\tilde{\nu}^{-1}} Z). \end{aligned}$$

If K' has interior points, then Minkowski's theorem (see [7], Theorem 7.1.2) implies that there is a unique convex body $T_{\tilde{\nu}} K'$ whose second area measure is given by $\int_{SO_3} S_2(\vartheta K', \cdot) \, d\tilde{\nu}(\vartheta)$. It follows from (3) that

$$V(K', K', R_{\tilde{\nu}^{-1}} Z) = V(T_{\tilde{\nu}} K', T_{\tilde{\nu}} K', Z).$$

By using the concept of mixed bodies (see, e.g., [7], p. 396, Note 7), the above formula could be extended to higher dimensions in a less explicit form.

Our next result is a generalization of a special case of Theorem 2.

Theorem 4. *Let α be a measure on $\mathbb{R}^n$ which is homogeneous of degree n and has a density function with respect to Lebesgue measure that is continuous on $\mathbb{R}^n\backslash\{0\}$. Define $Z \in \mathcal{K}$ to be the convex body whose support function is $\int_{B^n} |\langle\cdot, x\rangle|\, d\alpha(x)$. Let $j \in \{1,\dots,n-1\}$ and let $K_{j+1},\dots,K_{n-1} \in \mathcal{K}$. Let $a > 0$. Then we have*

$$\lim_{r\to\infty} \frac{1}{r^{n+1}} \int\limits_{rB^n} S(K \vee (K'+x)[j], K_{j+1},\dots,K_{n-1},\omega)\, d\alpha(x)$$
$$= \frac{1}{2}\sum_{i=0}^{j-1} S(K[i], K'[j-i-1], K_{j+1},\dots,K_{n-1}, Z, \omega),$$

uniformly for all $\omega \in \mathcal{B}(S^{n-1})$ and all $K, K' \in \mathcal{K}$ contained in the ball aB^n.

Remark 1. The proof of Theorem 4 can be extended to give an iterated version. Let $j \in \{1,\dots,n-1\}$, $k \in \{1,\dots,j\}$, and let $K_0,\dots,K_k$, $K_{j+1},\dots,K_{n-1} \in \mathcal{K}$. Define a function $g : (\mathbb{R}^n)^k \times \mathcal{B}(S^{n-1}) \to \mathbb{R}$ by

$$g(x_1,\dots,x_k,\omega) := S(K_0 \vee (K_1+x_1) \vee \cdots \vee (K_k+x_k)[j], K_{j+1},\dots,K_{n-1},\omega).$$

For $i \in \{1,\dots,k\}$, let α_i be a measure as in Theorem 4 and define $Z_i \in \mathcal{K}$ by $h_{Z_i} = \int_{B^n} |\langle\cdot,x\rangle|\, d\alpha_i(x)$. Then we have

$$\lim_{r\to\infty} \frac{1}{r^{k(n+1)}} \int\limits_{rB^n} \cdots \int\limits_{rB^n} g(x_1,\dots,x_k,\omega)\, d\alpha_1(x_1)\cdots d\alpha_k(x_k)$$
$$= \frac{1}{2^k} \sum_{\substack{r_0,\dots,r_k=0 \\ r_0+\cdots+r_k=j-k}}^{j-k} S(K_0[r_0],\dots,K_k[r_k], K_{j+1},\dots,K_{n-1}, Z_1,\dots,Z_k,\omega),$$

uniformly for all $\omega \in \mathcal{B}(S^{n-1})$ and all $K_0,\dots,K_k \in \mathcal{K}$ which are contained in a fixed ball.

Remark 2. The proof of Theorem 4 can also be modified to show the following variant. See Gardner [2] for the notions we use in this Remark. Let T be compact with interior points and star shaped with respect to 0, and let the radial function of T be continuous. Let ΓT be the centroid body of T, i.e. the convex body with support function $\lambda^n(T)^{-1} \int_T |\langle\cdot,x\rangle|\, d\lambda^n(x)$, where λ^n is Lebesgue measure on $\mathbb{R}^n$. Then we have

$$\lim_{r\to\infty} \frac{1}{r\lambda^n(rT)} \int\limits_{rT} S(K \vee (K'+x)[j], K_{j+1},\dots,K_{n-1},\omega)\, d\lambda^n(x)$$
$$= \frac{1}{2}\sum_{i=0}^{j-1} S(K[i], K'[j-i-1], K_{j+1},\dots,K_{n-1}, \Gamma T, \omega),$$

uniformly for all $\omega \in \mathcal{B}(S^{n-1})$ and all $K, K' \in \mathcal{K}$ which are contained in a fixed ball.

Our next theorem states a kinematic version of Theorem 4. If $g = g_{x,\vartheta} \in G_n$, we let $g_0 := \vartheta \in SO_n$ be the rotational part of g.

Theorem 5. *Under the assumptions of Theorem 4, we have*

$$\lim_{r\to\infty} \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n : gK' \subset rB^n\}} S(K \vee gK'[j], K_{j+1},\dots,K_{n-1}, \omega \cap g_0\omega')\, d\mu_\alpha(g)$$
$$= \frac{1}{2n\kappa_n} \sum_{i=0}^{j-1} S(K[i], B^n[j-i-1], K_{j+1},\dots,K_{n-1}, Z, \omega)\; S_{j-i-1}(K',\omega'),$$

uniformly for all $\omega, \omega' \in \mathcal{B}(S^{n-1})$ *and all* $K, K' \in \mathcal{K}$ *contained in* aB^n.

Remark. If we put $\mu := \mu_{\lambda^n}$, then μ is a Haar measure on G_n. A special case of Theorem 5 now reads

$$\lim_{r\to\infty} \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n : gK' \subset rB^n\}} S_j(K \vee gK', \omega \cap g_0\omega')\, d\mu(g) = \frac{\kappa_{n-1}}{(n+1)n\kappa_n} \sum_{i=0}^{j-1} S_i(K,\omega) S_{j-i-1}(K',\omega').$$

This formula can be considered as the analog of a special case of (2) in Euclidean space.

We conclude this section with some concise comments on a possible extension of the formula stated in the last Remark from the area measures to the support measures (or generalized curvature measures) of convex bodies. To define these measures, let $K \in \mathcal{K}$ and let Nor$K$ be the set of all support elements of K, i.e. the set of all pairs $(x, u) \in \mathbb{R}^n \times S^{n-1}$ where x is in the boundary bd$K$ of K and u is an outer unit normal vector of K at x. The support measures $\Theta_j(K, \cdot)$, $j \in \{0, \ldots, n-1\}$, are the unique measures on $\mathbb{R}^n \times S^{n-1}$ which are concentrated on Nor$K$ and satisfy the Steiner-type relation

$$\int\limits_{\mathbb{R}^n\setminus K} f\, d\lambda^n = \sum_{j=0}^{n-1} \binom{n-1}{j} \int\limits_0^\infty \int\limits_{\mathbb{R}^n\times S^{n-1}} t^{n-j-1} f(x+tu)\, d\Theta_j(K,(x,u))dt$$

for all Lebesgue integrable $f : \mathbb{R}^n \to \mathbb{R}$; compare Schneider [7], Theorem 4.2.1. We have $S_j(K, \omega) = \Theta_j(K, \mathbb{R}^n \times \omega)$ for all Borel sets $\omega \subset S^{n-1}$. We assume the measures $\Theta_j(K, \cdot)$ to be complete.

Next we define a law of composition between two subsets of $\mathbb{R}^n \times S^{n-1}$ which is adapted to the convex hull operation for pairs of convex bodies. For $\eta, \eta' \subset \mathbb{R}^n \times S^{n-1}$ and $g \in G_n$ we let

$$\begin{aligned} \eta \vee g\eta' \ := \ \big\{(x,u) \in \mathbb{R}^n \times S^{n-1} \ &: \ x \in [x_1, x_2] \text{ with } x_1, x_2 \in \mathbb{R}^n,\ (x_1, u) \in \eta, \\ &\quad (g^{-1}x_2, g_0^{-1}u) \in \eta'\big\}, \end{aligned}$$

where $[x_1, x_2]$ is the closed line segment with endpoints x_1, x_2. Again, we put $\mu := \mu_{\lambda^n}$. We want to state the following conjecture:

Conjecture 1. *Let* $j \in \{1, \ldots, n-1\}$ *and let* $a > 0$. *Then we have*

$$\lim_{r\to\infty} \frac{1}{r^{n+1}} \int\limits_{\{g\in G_n : gK' \subset rB^n\}} \Theta_j(K \vee gK', \eta \vee g\eta')\, d\mu(g) = \frac{\kappa_{n-1}}{(n+1)n\kappa_n} \sum_{i=0}^{j-1} \Theta_i(K,\eta)\ \Theta_{j-i-1}(K',\eta'),$$

uniformly for all $K, K' \in \mathcal{K}$ *contained in* aB^n *and for all Borel sets* $\eta \subset$ Nor$K$, $\eta' \subset$ NorK'.

The methods of the present paper, those of the work [4], and an application of Schneider's local rotation sum formula (see [7], Theorem 4.5.9) show that Conjecture 1 is true provided the following assertion about the boundary structure of convex bodies can be shown to be true.

Conjecture 2. *Let* $K, K' \in \mathcal{K}$. *Then for* μ*-a.e.* $g \in G_n$ *we have the following. For every* $x \in (\mathrm{bd}(K \vee gK'))\setminus(\mathrm{bd}K \cup \mathrm{bd}gK')$ *there are* unique *points* $y \in \mathrm{bd}K$, $z \in \mathrm{bd}gK'$ *such that* x *lies on the line segment* $[y, z]$.

This second conjecture was stated in a different but equivalent form in [3], p. 113. The argument sketched in Section 2 of [4] shows that Conjecture 2 is true in dimension $n = 3$ (the case $n = 2$ can easily be verified). Furthermore, a dualization of Theorem 2.2 in [4] can be used to show that Conjecture 2 is true in general dimensions if one of the bodies K, K' is a polytope. *Thus we can conclude that Conjecture 1 is true if* $n \le 3$ *and in general dimensions if one of the bodies* K, K' *is restricted to the class of all polytopes.*

Theorem 2.2 in [4], mentioned above, relies on a result of Zalgaller [11]. If Conjecture 2 turns out to be true, it will be a generalization of a result of Ivanov [5], who, like Zalgaller, used the methods developed by Ewald et al. [1]. We do not know whether these methods can also be applied to prove Conjecture 2.

3 Proofs of Theorems 2 and 3

We will use the following two lemmas. The first one is known, see Schneider & Weil [8], p. 308, Eq. (9.7), but we will give an alternative proof. For $x_1, \ldots, x_k \in \mathbb{R}^n$, we denote by $D_k(x_1, \ldots, x_k)$ the k-dimensional volume of the parallelepiped spanned by these vectors. If $K \in \mathcal{K}$ and if L is a linear subspace, we denote by $K|L$ the orthogonal projection of K onto L. In the proofs of Theorems 2 and 4 we will use only the case $k = 1$ of this lemma. However, if one wants to modify the proof of Theorem 4 to show the iterated extension, one may utilize the second part of Lemma 1 in the version stated below.

Lemma 1. *Let $k \in \{1, \ldots, n\}$. For $i \in \{1, \ldots, k\}$, let ρ_i be a finite measure on S^{n-1}, and let $Z_i \in \mathcal{K}$ be the convex body with support function $\int_{S^{n-1}} |\langle \cdot, v\rangle| \, d\rho_i(v)$. Let $K_1, \ldots, K_{n-k} \in \mathcal{K}$. Then we have*

$$n(n-1)\cdots(n-k+1)\; V(K_1, \ldots, K_{n-k}, Z_1, \ldots, Z_k)$$

$$= 2^k \int_{S^{n-1}} \cdots \int_{S^{n-1}} D_k(u_1, \ldots, u_k)\, V^L(K_1|L, \ldots, K_{n-k}|L)\, d\rho_1(u_1) \cdots d\rho_k(u_k),$$

where L is the $(n-k)$-subspace orthogonal to $u_1, \ldots, u_k$ (provided that $D_k(u_1, \ldots, u_k) > 0$) and V^L denotes the mixed volume functional in L (if $k = n$, then $V^L \equiv 1$). If $k \le n-1$ and $\omega \in \mathcal{B}(S^{n-1})$, then

$$(n-1)\cdots(n-k)\; S(K_1, \ldots, K_{n-k-1}, Z_1, \ldots, Z_k, \omega)$$

$$= 2^k \int_{S^{n-1}} \cdots \int_{S^{n-1}} D_k(u_1, \ldots, u_k)\, S^L(K_1|L, \ldots, K_{n-k-1}|L, \omega \cap L)\, d\rho_1(u_1) \cdots d\rho_k(u_k),$$

where S^L is the mixed area measure in the subspace L (if $k = n-1$, then S^L is the counting measure on $S^{n-1} \cap L$).

Proof. The first equation can be shown by a straightforward induction on k, where (3), the Fubini theorem, and the relation

$$V^H(K_1|H, \ldots, K_{n-1}|H) = \frac{1}{2} \int_{S^{n-1}} |\langle u, v\rangle| \, dS(K_1, \ldots, K_{n-1}, v)$$

are used. Here $K_1, \ldots, K_{n-1}$ are arbitrary convex bodies and H is a hyperplane through 0 with unit normal vector u; see Schneider [7], p. 296, Eq. (5.3.32).

In the case $k \le n-1$, a further application of (3) gives

$$\begin{aligned}
&(n-1)\cdots(n-k) \int_{S^{n-1}} h_{K_{n-k}} \, dS(K_1, \ldots, K_{n-k-1}, Z_1, \ldots, Z_k, \cdot) \\
&\quad = 2^k \int_{S^{n-1}} \cdots \int_{S^{n-1}} D_k(u_1, \ldots, u_k) \int_{S^{n-1}\cap L} h_{K_{n-k}} \\
&\qquad\qquad dS^L(K_1|L, \ldots, K_{n-k-1}|L, \cdot) d\rho_1(u_1) \cdots d\rho_k(u_k),
\end{aligned}$$

and since the set of all differences of support functions is dense in $C(S^{n-1})$ (see [7], Lemma 1.7.9), the function $h_{K_{n-k}}$ can be replaced by an arbitrary continuous function. Now the uniqueness statement in Riesz' representation theorem shows that it can also be replaced by the indicator function of a Borel set $\omega \subset S^{n-1}$; this gives the second equation of Lemma 1. ■

Below we will use the following notations. $\mathcal{P}$ denotes the class of all convex polytopes. For $K \in \mathcal{K}$ and $u \in S^{n-1}$, $F(K, u) := \{x \in K : \langle x, u\rangle = h_K(u)\}$ is the support set of K at u. The volume functional on $\mathcal{K}$ is denoted by V_n. For $K, K' \in \mathcal{K}$, $K + K'$ is the Minkowski sum of these two bodies. The dimension of a convex body K is denoted by $\dim K$.

Lemma 2. *Let $P, P' \in \mathcal{P}$ and let $U := \{u \in S^{n-1} : \dim F(P+P',u) = n-1\}$. Then*

$$2\, V_n(P \vee P') = V_n(P) + V_n(P') + \sum_{u \in U} V_n(F(P,u) \vee F(P',u)).$$

Proof. Let $U_1 := \{u \in U : h_P(u) > h_{P'}(u)\}$. We have

$$(F(P,u) \vee F(P',u)) \backslash F(P',u) = \bigcup_{0<s\le 1} F(sP + (1-s)P', u)$$

for all $u \in U_1$ and

$$(P \vee P') \backslash P' = \bigcup_{0<s\le 1} \bigcup_{u \in U_1} F(sP + (1-s)P', u),$$

where the union over s is disjoint. It follows that

$$(P \vee P') \backslash P' = \bigcup_{u \in U_1} (F(P,u) \vee F(P',u)) \backslash F(P',u).$$

Furthermore, for $u, u' \in U_1$, $u \neq u'$, the interiors of the sets $F(P,u) \vee F(P',u)$ and $F(P,u') \vee F(P',u')$ are disjoint, since otherwise, for some $s \in (0,1)$, the relative interiors of the facets $F(sP+(1-s)P',u)$ and $F(sP+(1-s)P',u')$ of $sP+(1-s)P'$ would have non-empty intersection, which is impossible as these facets do not coincide. We conclude

$$V_n((P \vee P') \backslash P') = \sum_{u \in U_1} V_n(F(P,u) \vee F(P',u))$$

and analogously we get $V_n((P \vee P') \backslash P) = \sum_{u \in U_2} V_n(F(P,u) \vee F(P',u))$ where $U_2 := \{u \in U : h_P(u) < h_{P'}(u)\}$. If $u \in U$ with $h_P(u) = h_{P'}(u)$, then clearly $\dim(F(P,u) \vee F(P',u)) \le n-1$. Now the result follows. ∎

In the proofs of Theorems 2 and 4 we will use the following notations. For a vector $v \neq 0$, the hyperplane through 0 and orthogonal to v is denoted by $v^{\perp}$. For $K \in \mathcal{K}$ and $i \in \{0, \ldots, n\}$, let $V_i(K)$ be the i-th intrinsic volume. So if $\dim K = n$, then $2V_{n-1}(K)$ is the surface area of K and if $\dim K = i$, then $V_i(K)$ is the i-dimensional volume of K.

Proof of Theorem 2. Let the assumptions of Theorem 2 be given. We begin by showing that it suffices to establish the relation

$$\lim_{r\to\infty} \frac{1}{r^{\delta+1}} \int_{rB^n} V_n(K \vee (K'+x))\, d\alpha(x) = \frac{n}{2} \int_0^1 V(sK + (1-s)K'[n-1], Z)\, ds, \tag{4}$$

where the limit is uniform for all $K, K' \in \mathcal{K}$ with $K, K' \subset aB^n$.

So let us assume that this is true. Since $h_{K \vee K'} = \max\{h_K, h_{K'}\}$ and $h_{K+K'} = h_K + h_{K'}$ for all $K, K' \in \mathcal{K}$, we have $h_{(K\vee K')+K''} = h_{(K+K'')\vee(K'+K'')}$ and thus $(K \vee K') + K'' = (K+K'') \vee (K'+K'')$ for all $K, K', K'' \in \mathcal{K}$. Hence if we replace K and K' in (4) by $K + tK_n$ and $K' + tK_n$, $t \ge 0$, the additivity properties of the mixed volume imply that

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r^{\delta+1}}\sum_{i=0}^{n}\binom{n}{i}t^i\int\limits_{rB^n} V(K\vee(K'+x)[n-i],K_n[i])\,d\alpha(x)\\
&= \lim_{r\to\infty}\frac{1}{r^{\delta+1}}\int\limits_{rB^n} V_n((K\vee(K'+x))+tK_n)\,d\alpha(x)\\
&= \lim_{r\to\infty}\frac{1}{r^{\delta+1}}\int\limits_{rB^n} V_n((K+tK_n)\vee(K'+tK_n+x))\,d\alpha(x)\\
&= \frac{n}{2}\int\limits_0^1 V(s(K+tK_n)+(1-s)(K'+tK_n)[n-1],Z)\,ds\\
&= \frac{n}{2}\int\limits_0^1 V(sK+(1-s)K'+tK_n[n-1],Z)\,ds\\
&= \frac{n}{2}\sum_{i=0}^{n-1}\binom{n-1}{i}t^i\int\limits_0^1 V(sK+(1-s)K'[n-i-1],K_n[i],Z)\,ds
\end{aligned}$$

where the limits are uniform in $K,K'\subset aB^n$. Comparison of the coefficients of t gives

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r^{\delta+1}}\int\limits_{rB^n} V(K\vee(K'+x)[n-1],K_n)\,d\alpha(x)\\
&\qquad=\frac{n-1}{2}\int\limits_0^1 V(sK+(1-s)K'[n-2],K_n,Z)\,ds,
\end{aligned}$$

and repetition of this argument yields

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r^{\delta+1}}\int\limits_{rB^n} V(K\vee(K'+x)[j],K_{j+1},\dots,K_n)\,d\alpha(x)\\
&= \frac{j}{2}\int\limits_0^1 V(sK+(1-s)K'[j-1],K_{j+1},\dots,K_n,Z)\,ds\\
&= \frac{j}{2}\sum_{i=0}^{j-1}\binom{j-1}{i}\int\limits_0^1 s^i(1-s)^{j-i-1}ds\;V(K[i],K'[j-i-1],K_{j+1},\dots,K_n,Z)\\
&= \frac{1}{2}\sum_{i=0}^{j-1}V(K[i],K'[j-i-1],K_{j+1},\dots,K_n,Z),
\end{aligned}$$

uniformly for $K,K'\subset aB^n$, where we used $j\binom{j-1}{i}\int_0^1 s^i(1-s)^{j-i-1}\,ds=1$, which can be shown by i-fold integration by parts. Now the first formula of Theorem 2 is deduced from (4).

Using (3), the above argument shows that (4) implies

$$\begin{aligned}
&\lim_{r\to\infty}\frac{1}{r^{\delta+1}}\int\limits_{rB^n}\int\limits_{S^{n-1}} h_{K_n}\,dS(K\vee(K'+x)[j],K_{j+1},\dots,K_{n-1},\cdot)\,d\alpha(x)\\
&\qquad=\frac{1}{2}\sum_{i=0}^{j-1}\int\limits_{S^{n-1}} h_{K_n}\,dS(K[i],K'[j-i-1],K_{j+1},\dots,K_{n-1},Z,\cdot),
\end{aligned}$$

uniformly for $K, K' \subset aB^n$. As the continuous function f can be approximated arbitrarily closely, in the maximum norm, by differences $h_{K_n} - h_{\tilde{K}_n}$ for $K_n, \tilde{K}_n \in \mathcal{K}$ (see [7], Lemma 1.7.9), we can use the triangle inequality to deduce that

$$\lim_{r\to\infty} \frac{1}{r^{\delta+1}} \int\limits_{rB^n} \int\limits_{S^{n-1}} f\, dS(K \vee (K'+x)[j], K_{j+1}, \dots, K_{n-1}, \cdot)\, d\alpha(x)$$

$$= \frac{1}{2} \sum_{i=0}^{j-1} \int\limits_{S^{n-1}} f\, dS(K[i], K'[j-i-1], K_{j+1}, \dots, K_{n-1}, Z, \cdot),$$

uniformly for all $K, K' \subset aB^n$. Now also the second relation of Theorem 2 is reduced to (4).

So our aim is to prove the relation (4). Since the right side of (4), as well as the left side for any fixed $r > 0$, is continuous in K, K', it suffices to prove that for all $a > 0$ there is a $c > 0$ such that

$$\left| \frac{1}{r^{\delta+1}} \int\limits_{rB^n} V_n(P \vee (P'+x))\, d\alpha(x) - \frac{n}{2} \int\limits_0^1 V(sP + (1-s)P'[n-1], Z)\, ds \right| \le \frac{c}{r}$$

for all $r > 0$ and all polytopes $P, P' \in \mathcal{P}$ which are contained in the ball aB^n.

We will need a polar decomposition of the measure α, which can be achieved as follows. Define a finite measure ρ on S^{n-1} by

$$\rho(\omega) := \delta\, \alpha(\{tx \in \mathbb{R}^n : 0 \le t \le 1, x \in \omega\}), \qquad \omega \in \mathcal{B}(S^{n-1}),$$

and define a measure $\tilde{\alpha}$ on $\mathbb{R}^n$ by

$$\tilde{\alpha}(A) := \int\limits_0^\infty \int\limits_{S^{n-1}} t^{\delta-1} \mathbf{1}_A(tv)\, d\rho(v)\, dt,$$

where $\mathbf{1}_A$ denotes the indicator function of the Borel set $A \in \mathcal{B}(\mathbb{R}^n)$. It follows from the local finiteness of α that the spheres tS^{n-1}, $t \ge 0$, have α-measure zero. Therefore α and $\tilde{\alpha}$ coincide on the sets $\{tx \in \mathbb{R}^n : t_1 \le t \le t_2, x \in \omega\}$, where $0 \le t_1 < t_2$ and $\omega \in \mathcal{B}(S^{n-1})$. Since $\mathbb{R}^n$ is the union of an increasing sequence of such sets and since they are closed under intersections and generate $\mathcal{B}(\mathbb{R}^n)$, it follows that $\tilde{\alpha} = \alpha$. We therefore have

$$\int\limits_{\mathbb{R}^n} g\, d\alpha = \int\limits_0^\infty \int\limits_{S^{n-1}} t^{\delta-1} g(tv)\, d\rho(v) dt$$

for all measurable functions $g \ge 0$. In particular

$$\int\limits_{B^n} |\langle \cdot, x\rangle|\, d\alpha(x) = \int\limits_0^1 \int\limits_{S^{n-1}} t^{\delta-1} |\langle \cdot, tv\rangle|\, d\rho(v) dt = \frac{1}{\delta+1} \int\limits_{S^{n-1}} |\langle \cdot, v\rangle|\, d\rho(v),$$

so if $Z' \in \mathcal{K}$ is the body with support function $\int_{S^{n-1}} |\langle \cdot, v\rangle|\, d\rho(v)$, then $Z = \frac{1}{\delta+1} Z'$.

For the following, let $a > 0$ and let $P, P' \in \mathcal{P}$ with $P, P' \subset aB^n$. Let $r > 0$. Define

$$U := \{u \in S^{n-1} : \dim F(P + P', u) = n-1\}.$$

According to Lemma 1 we have

$$
\begin{aligned}
&\frac{n}{2}\int_0^1 V(sP+(1-s)P'[n-1],Z)\,ds\\
&= \frac{n}{2(\delta+1)}\int_0^1 V(sP+(1-s)P'[n-1],Z')\,ds\\
&= \frac{1}{\delta+1}\int_0^1\int_{S^{n-1}} V_{n-1}((sP+(1-s)P')|v^{\perp})\,d\rho(v)ds\\
&= \frac{1}{2(\delta+1)}\int_{S^{n-1}}\int_0^1 \sum_{u\in U} V_{n-1}(F(sP+(1-s)P',u)|v^{\perp})\,dsd\rho(v)\\
&= \frac{1}{2(\delta+1)}\int_{S^{n-1}}\int_0^1 \sum_{u\in U} |\langle u,v\rangle| V_{n-1}(F(sP+(1-s)P',u))\,dsd\rho(v)\\
&= \frac{1}{2r^{\delta+1}}\int_0^r\int_{S^{n-1}}\int_0^1 t^{\delta-1}\sum_{u\in U} |\langle u,tv\rangle| V_{n-1}(F(sP+(1-s)P',u))\,dsd\rho(v)dt. \qquad (5)
\end{aligned}
$$

On the other hand, we infer from Lemma 2 that

$$
\begin{aligned}
&\frac{1}{r^{\delta+1}}\int_{rB^n} V_n(P\vee(P'+x))\,d\alpha(x)\\
&= \frac{1}{2r^{\delta+1}}\int_{rB^n}\Big(V_n(P)+V_n(P')+\sum_{u\in U}V_n(F(P,u)\vee(F(P',u)+x))\Big)\,d\alpha(x)\\
&= \frac{\alpha(B^n)(V_n(P)+V_n(P'))}{2r}\\
&\qquad + \frac{1}{2r^{\delta+1}}\int_0^r\int_{S^{n-1}} t^{\delta-1}\sum_{u\in U}V_n(F(P,u)\vee(F(P',u)+tv))\,d\rho(v)dt. \qquad (6)
\end{aligned}
$$

Let us show the following assertion: If K_1 and K_2 are convex bodies which are contained in a single hyperplane, then

$$V_n(K_1\vee(K_2+x)) = V_n(K_1|x^{\perp}\vee(K_2|x^{\perp}+x))$$

for all $x\neq 0$. We can assume $\dim(K_1+K_2)=n-1$, $0\in \operatorname{aff}(K_1+K_2)$ and $x\notin\operatorname{aff}(K_1+K_2)$, where aff denotes the affine hull operation. Choose a basis $\{y_1,\dots,y_{n-1}\}$ in the linear subspace $\operatorname{aff}(K_1+K_2)$. The linear map that fixes x and maps each y_i to $y_i|x^{\perp}$ has determinant one and maps $K_1\vee(K_2+x)$ onto $K_1|x^{\perp}\vee(K_2|x^{\perp}+x)$. This shows the above equation.

For each $u\in U$, there is a $y_u\in 2aB^n$ such that $F(P,u)$ and $F(P',u)-y_u$ are contained in a single hyperplane. What we have shown above implies that

$$
\begin{aligned}
V_n(F(P,u)\vee(F(P',u)+tv)) &= V_n(F(P,u)\vee(F(P',u)-y_u+tv+y_u))\\
&= V_n(F(P,u)|(tv+y_u)^{\perp}\vee((F(P',u)-y_u)|(tv+y_u)^{\perp}+tv+y_u))
\end{aligned}
$$

for all $u \in U$, $v \in S^{n-1}$, $t \geq 0$ with $tv \neq -y_u$. Now Cavalieri's principle shows that

$$\begin{aligned}
& V_n(F(P,u) \vee (F(P',u)+tv)) \\
= & \int_0^{\|tv+y_u\|} V_{n-1}\Big(\Big(\frac{s}{\|tv+y_u\|}F(P,u) + \frac{\|tv+y_u\|-s}{\|tv+y_u\|}(F(P',u)-y_u)\Big)\Big|(tv+y_u)^{\perp}\Big)\,ds \\
= & \|tv+y_u\| \int_0^1 V_{n-1}((sF(P,u)+(1-s)F(P',u))|(tv+y_u)^{\perp})\,ds \\
= & |\langle u, tv+y_u\rangle| \int_0^1 V_{n-1}(sF(P,u)+(1-s)F(P',u))\,ds \\
= & |\langle u, tv+y_u\rangle| \int_0^1 V_{n-1}(F(sP+(1-s)P',u))\,ds \qquad (7)
\end{aligned}$$

for all $u \in U$, $v \in S^{n-1}$, $t \geq 0$ with $tv \neq -y_u$. Both the first and the last term equal zero if $tv = -y_u$. Because of

$$\Big||\langle u, tv+y_u\rangle| - |\langle u, tv\rangle|\Big| \leq |\langle u, y_u\rangle| \leq \|y_u\| \leq 2a$$

for all $u \in U$, $v \in S^{n-1}$, $t \geq 0$, and since $V_{n-1}(sP+(1-s)P') \leq V_{n-1}(2aB^n)$ for $0 \leq s \leq 1$, we get from (5), (6), and (7)

$$\begin{aligned}
& \left|\frac{1}{r^{\delta+1}}\int_{rB^n} V_n(P \vee (P'+x))\,d\alpha(x) - \frac{n}{2}\int_0^1 V(sP+(1-s)P'[n-1],Z)\,ds\right| \\
\leq & \frac{c_1}{r} + \frac{1}{2r^{\delta+1}}\int_0^r \int_{S^{n-1}} \int_0^1 t^{\delta-1} \sum_{u\in U} V_{n-1}(F(sP+(1-s)P',u)) \\
& \qquad \Big||\langle u, tv+y_u\rangle| - |\langle u, tv\rangle|\Big|\,ds d\rho(v) dt \\
\leq & \frac{c_1}{r} + \frac{c_2}{r}\int_0^1 V_{n-1}(sP+(1-s)P')\,ds \leq \frac{c}{r}
\end{aligned}$$

with suitable numbers $c, c_1, c_2 > 0$, which depend on a, but not on P or P'. This is what we wanted to show. ∎

In the following we indicate how the iterated formula stated in the Remark after Theorem 2 can be obtained. It is sufficient to show that for all $a > 0$ there is a $c > 0$ so that

$$\begin{aligned}
& \Bigg|\frac{1}{r^{\delta_1+\cdots+\delta_k+k}}\int_{rB^n}\cdots\int_{rB^n} V_n(P_0 \vee (P_1+x_1) \vee \cdots \vee (P_k+x_k))\,d\alpha_1(x_1)\cdots d\alpha_k(x_k) \\
& - \frac{1}{2^k}\sum_{\substack{r_0,\ldots,r_k=0 \\ r_0+\cdots+r_k=n-k}}^{n-k} V(P_0[r_0],\ldots,P_k[r_k],Z_1,\ldots,Z_k)\Bigg| \leq \frac{c}{r}
\end{aligned}$$

for all polytopes $P_0, \ldots, P_k \subset aB^n$ and all $r \geq 1$. Let $k > 1$ and assume that the corresponding assertion is true for all smaller positive integers. Let $a > 0$, $r \geq 1$, and let $P_0, \ldots, P_k \subset aB^n$ be polytopes. For $x_1, \ldots, x_{k-1} \in \mathbb{R}^n$

let $U(x_1,\dots,x_{k-1}) := \{u \in S^{n-1} : \dim F((P_0 \vee (P_1+x_1) \vee \cdots \vee (P_{k-1}+x_{k-1}))+P_k, u) = n-1\}$. The inductive hypothesis and Lemma 2 imply that

$$\begin{aligned}
&\frac{1}{r^{\delta_1+\cdots+\delta_k+k}}\Bigg| \int\limits_{rB^n} \cdots \int\limits_{rB^n} V_n(P_0 \vee (P_1+x_1) \vee \cdots \vee (P_k+x_k))\, d\alpha_1(x_1)\cdots d\alpha_k(x_k) \\
&\quad - \frac{1}{2}\int\limits_{rB^n} \cdots \int\limits_{rB^n} \sum_{u\in U(x_1,\dots,x_{k-1})} V_n(F(P_0 \vee (P_1+x_1) \vee \cdots \vee (P_{k-1}+x_{k-1}), u) \vee (F(P_k,u)+x_k)) \\
&\quad d\alpha_1(x_1)\cdots d\alpha_k(x_k)\Bigg| \le \frac{c_1}{r},
\end{aligned}$$

where here and below $c_1,\dots,c_4$ denote suitable positive numbers which are independent of $P_0,\dots,P_k$. It can be shown as in the proof of Theorem 2 that

$$\begin{aligned}
&\Bigg| \frac{1}{r^{\delta_k+1}} \int\limits_{rB^n} \sum_{u\in U(x_1,\dots,x_{k-1})} V_n(F(P_0 \vee \cdots \vee (P_{k-1}+x_{k-1}), u) \vee (F(P_k,u)+x_k))\, d\alpha_k(x_k) \\
&\quad - \frac{1}{\delta_k+1} \int\limits_{S^{n-1}} \int\limits_0^1 V_{n-1}((s(P_0 \vee (P_1+x_1) \vee \cdots \vee (P_{k-1}+x_{k-1})) + (1-s)P_k)|v^\perp)\, ds d\rho_k(v)\Bigg| \\
\le\ & \frac{c_2}{r}\, V_{n-1}(2aB^n \vee (2aB^n+x_1) \vee \cdots \vee (2aB^n + x_{k-1})) \le c_3 r^{k-2}
\end{aligned}$$

for all $x_1,\dots,x_{k-1} \in rB^n$, where ρ_k is the measure on S^{n-1} corresponding to α_k (see the above proof). We can now apply Lemma 1 to show that

$$\begin{aligned}
&\Bigg| \frac{1}{r^{\delta_1+\cdots+\delta_k+k}} \int\limits_{rB^n} \cdots \int\limits_{rB^n} V_n(P_0 \vee (P_1+x_1) \vee \cdots \vee (P_k+x_k))\, d\alpha_1(x_1)\cdots d\alpha_k(x_k) \\
&\quad - \frac{1}{2r^{\delta_1+\cdots+\delta_{k-1}+k-1}} \int\limits_{rB^n} \cdots \int\limits_{rB^n} \\
&\quad \sum_{i=0}^{n-1} V(P_0 \vee \cdots \vee (P_{k-1}+x_{k-1})[i], P_k[n-i-1], Z_k)\, d\alpha_1(x_1)\cdots d\alpha_{k-1}(x_{k-1})\Bigg| \le \frac{c_4}{r}.
\end{aligned}$$

Now the assertion can be deduced from the inductive hypothesis.

Proof of Theorem 3. Let the assumptions of Theorem 2 be given, and let $K, K' \in \mathcal{K}$ with $K, K' \subset aB^n$. For all $\vartheta \in SO_n$, $r > 0$, and $x \in \mathbb{R}^n$, we have the implications

$$\begin{aligned}
x \in rB^n &\implies \vartheta K' + x \subset (r+a)B^n, \\
\vartheta K' + x \subset (r+a)B^n &\implies x \in (r+2a)B^n.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{r^{\delta+1}} \int\limits_{rB^n} \int\limits_{SO_n} V(K \vee (\vartheta K' + x)[j], K_{j+1},\dots,K_n)\, d\nu(\vartheta) d\alpha(x) \\
&\le \frac{1}{r^{\delta+1}} \int\limits_{\{g\in G_n : gK' \subset (r+a)B^n\}} V(K \vee gK'[j], K_{j+1},\dots,K_n)\, d\mu_\alpha(g) \\
&\le \frac{1}{r^{\delta+1}} \int\limits_{(r+2a)B^n} \int\limits_{SO_n} V(K \vee (\vartheta K' + x)[j], K_{j+1},\dots,K_n)\, d\nu(\vartheta) d\alpha(x).
\end{aligned}$$

Equation (5.3.25) on p. 294 in Schneider [7] gives

$$\begin{aligned}&\int\limits_{SO_n} V(K[i],\vartheta K'[j-i-1],K_{j+1},\dots,K_n,Z)\,d\nu(\vartheta)\\&\quad=\frac{1}{\kappa_n}\,V(K[i],B^n[j-i-1],K_{j+1},\dots,K_n,Z)\;V(K'[j-i-1],B^n[n-j+i+1]),\end{aligned}$$

$i\in\{0,\dots,j-1\}$. We now deduce from Fubini's theorem, Theorem 2, and the dominated convergence theorem that both the first and the third term in the above inequalities converge for $r\to\infty$ to

$$\frac{1}{2\kappa_n}\sum_{i=0}^{j-1}V(K[i],B^n[j-i-1],K_{j+1},\dots,K_n,Z)\;V(K'[j-i-1],B^n[n-j+i+1]),$$

uniformly for $K,K'\subset aB^n$. Thus the second term has the same limit, and Theorem 3 follows. ■

4 Proofs of Theorems 4 and 5

In the proof of Theorem 4 we will make use of the following fact. Let P be an n-dimensional polytope and define $\mathcal{F}_i(P)$, $i\in\{0,\dots,n-1\}$, to be the set of all i-dimensional faces of P. Then

$$S_{n-1}(P,\omega)=\sum_{F\in\mathcal{F}_{n-1}(P)}V_{n-1}(F)\,\mathbf{1}_\omega(u_F),\tag{8}$$

where u_F is the outer unit normal vector of P at F.

Proof of Theorem 4. Let the assumptions of Theorem 4 be given. The measurability of the integrand on the left hand side, for any fixed $r>0$, can be proved by standard methods of integration theory, see, e.g., Hilfssatz 7.2.2 in Schneider & Weil [9].

We will use the concept of the total variation norm $\|\mu\|$ of a finite signed measure μ on S^{n-1}. We have $\|\mu\|=\sup\{\int_{S^{n-1}}f\,d\mu : f\in C(S^{n-1}),\|f\|_\infty\le 1\}$, where $\|f\|_\infty$ denotes the maximum norm of f, and $\|\mu\|\le 2\sup_{\omega\in\mathcal{B}(S^{n-1})}|\mu(\omega)|\le 2\|\mu\|$. It can be seen as in the proof of Theorem 2 that it is sufficient to show that

$$\lim_{r\to\infty}\frac{1}{r^{n+1}}\int\limits_{rB^n}S_{n-1}(K\vee(K'+x),\cdot)\,d\alpha(x)=\frac{n-1}{2}\int\limits_0^1 S(sK+(1-s)K'[n-2],Z,\cdot)\,ds,$$

where the limit is in the sense of the total variation norm and uniform in $K,K'\subset aB^n$. Since the right side, as well as the left side for any fixed $r>0$, are weakly continuous in $K,K'\in\mathcal{K}$, it suffices to show that for all $\epsilon>0$ and all $a>0$, there is an $r_0>0$ such that

$$\begin{aligned}\Bigg|\frac{1}{r^{n+1}}\int\limits_{rB^n}&S_{n-1}(P\vee(P'+x),\omega)\,d\alpha(x)\\&-\frac{n-1}{2}\int\limits_0^1 S(sP+(1-s)P'[n-2],Z,\omega)\,ds\Bigg|\le\epsilon\end{aligned}\tag{9}$$

for all $r\ge r_0$, all $\omega\in\mathcal{B}(S^{n-1})$, and all n-dimensional polytopes $P,P'\in\mathcal{P}$ which are contained in aB^n. For the following, assume that ϵ and a are given, and let $P,P'\in\mathcal{P}$ be n-dimensional polytopes with $P,P'\subset aB^n$ and $\omega\in\mathcal{B}(S^{n-1})$.

Let $f : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}$ be the continuous density of α with respect to Lebesgue measure λ^n. Again as in the proof of Theorem 2, it follows that

$$\alpha(A) = \int_0^\infty \int_{S^{n-1}} t^{n-1} \mathbf{1}_A(tv) f(v)\, d\sigma(v) dt, \qquad A \in \mathcal{B}(\mathbb{R}^n),$$

where σ is spherical Lebesgue measure on S^{n-1}, and that $f(tv) = f(v)$ for all $v \in S^{n-1}, t > 0$. If $Z' \in \mathcal{K}$ has support function $\int_{S^{n-1}} |\langle \cdot, v\rangle| f(v)\, d\sigma(v)$, then $Z = \frac{1}{n+1} Z'$.

Let us define

$$\mathcal{F} := \{(F, F') \in \mathcal{F}^*(P) \times \mathcal{F}^*(P') : F + F' \in \mathcal{F}_{n-2}(P + P')\},$$

where $\mathcal{F}^*(P)$ is the set of all non-empty faces of P, and for $(F, F') \in \mathcal{F}$ let

$$\begin{aligned} A_{F,F'} := \{x \in \mathbb{R}^n \;&:\; F \vee (F' + x) \text{ is a facet of } P \vee (P' + x), \text{ and its outer} \\ &\quad\;\text{unit normal vector is contained in } \omega\}, \\ B_{F,F'} := \{v \in S^{n-1} \;&:\; (F + F')|v^\perp \text{ is a facet of } (P + P')|v^\perp, \text{ and its outer unit} \\ &\quad\;\text{normal vector (in the subspace } v^\perp) \text{ is contained in } \omega\} \end{aligned}$$

(we do not indicate the dependence on P, P' and ω, since these remain fixed in the argument below). Let further

$$\begin{aligned} C := \{x \in \mathbb{R}^n \;&:\; \text{for each facet } G \text{ of } P \vee (P' + x) \text{ which is not a facet of } P \\ &\quad\;\text{or } P' + x, \text{ we have } (G \cap P) + ((G - x) \cap P') \in \mathcal{F}_{n-2}(P + P')\}, \\ D := \{v \in S^{n-1} \;&:\; v \text{ is not parallel to a segment in the boundary of } P + P'\}. \end{aligned}$$

The following facts will be used below.

(a) For $x \in \mathbb{R}^n$, each facet of $P \vee (P' + x)$ which is not a facet of P or $P' + x$ is of the form $F \vee (F' + x)$, where $F \in \mathcal{F}^*(P)$, $F' \in \mathcal{F}^*(P')$.

(b) Let $F \in \mathcal{F}^*(P), F' \in \mathcal{F}^*(P')$. If the set $\{x \in \mathbb{R}^n : F \vee (F' + x) \in \mathcal{F}_{n-1}(P \vee (P' + x))\}$ has positive Lebesgue measure, then $(F, F') \in \mathcal{F}$.

(c) $\lambda^n(\mathbb{R}^n \backslash C) = 0$.

(d) If $x \in C$ and if $(F, F'), (\tilde{F}, \tilde{F}')$ are distinct members of $\mathcal{F}$ such that both $F \vee (F' + x)$ and $\tilde{F} \vee (\tilde{F}' + x)$ are facets of $P \vee (P' + x)$, then $F \vee (F' + x) \neq \tilde{F} \vee (\tilde{F}' + x)$.

(e) $\sigma(S^{n-1} \backslash D) = 0$.

(f) If $v \in D$ and if $(F, F'), (\tilde{F}, \tilde{F}')$ are distinct members of $\mathcal{F}$ such that both $(F + F')|v^\perp$ and $(\tilde{F} + \tilde{F}')|v^\perp$ are facets of $(P + P')|v^\perp$, then $(F + F')|v^\perp \neq (\tilde{F} + \tilde{F}')|v^\perp$.

(g) Let $(F, F') \in \mathcal{F}$, $y \in \mathbb{R}^n$ with $\dim \operatorname{aff}(F \cup (F' - y)) = n - 2$, $v \in D$, and $t > 0$. Then we have the equivalence

$$v \in B_{F,F'} \iff tv - y \in A_{F,F'}.$$

Let us briefly verify these assertions. For (a), let $x \in \mathbb{R}^n$ and let G be a facet of $P \vee (P' + x)$ which is not a facet of P or $P' + x$. Let $F := G \cap P$, $F' := (G - x) \cap P'$. Then $G = F \vee (F' + x)$ and $F \in \mathcal{F}^*(P)$, $F' \in \mathcal{F}^*(P')$. For (b), let $F \in \mathcal{F}^*(P)$, $F' \in \mathcal{F}^*(P')$ and let $M(F, F') := \{x \in \mathbb{R}^n : F \vee (F' + x) \in \mathcal{F}_{n-1}(P \vee (P' + x))\}$. If $\dim(F + F') < n - 2$, then $M(F, F') = \emptyset$. If $\dim(F + F') = n - 1$, then $M(F, F')$ is contained in a hyperplane. If $\dim(F + F') = n - 2$ and $F + F'$ is not a face of $P + P'$, then $M(F, F')$ is contained in the union of all $M(\tilde{F}, \tilde{F}')$ where $(\tilde{F}, \tilde{F}') \in \mathcal{F}^*(P) \times \mathcal{F}^*(P')$ with $\dim(\tilde{F} + \tilde{F}') = n - 1$. For (c), let $x \in \mathbb{R}^n \backslash C$. Then there is a $G \in \mathcal{F}_{n-1}(P \vee (P' + x)) \backslash (\mathcal{F}_{n-1}(P) \cup \mathcal{F}_{n-1}(P' + x))$ with $(G \cap P) + ((G - x) \cap P') \in \mathcal{F}_{n-1}(P + P')$. It follows from (b) that the set of all such x has Lebesgue measure zero. For (d), let $x \in C$ and let $G \in \mathcal{F}_{n-1}(P \vee (P' + x)) \backslash (\mathcal{F}_{n-1}(P) \cup \mathcal{F}_{n-1}(P' + x))$. Then $F := G \cap P$ and $F' := (G - x) \cap P'$ define the unique pair

$(F,F')\in\mathcal{F}$ with $G=F\vee(F'+x)$. This proves (d). For (e), note that $S^{n-1}\backslash D$ equals the union of finitely many great subspheres of S^{n-1}. For (f), let the assumptions be given and assume $(F+F')|v^\perp=(\tilde F+\tilde F')|v^\perp$. Then clearly $v\notin D$, a contradiction. For (g), let the assumptions be given. Clearly both conditions are equivalent to the existence of a common supporting hyperplane H of P and $P'-y$ which is parallel to v and satisfies $H\cap P=F$, $(H+y)\cap P'=F'$.

From (a) – (d), the fact that α is absolutely continuous with respect to λ^n, and (8), we get

$$\begin{aligned}&\frac{1}{r^{n+1}}\left|\int_{rB^n} S_{n-1}(P\vee(P'+x),\omega)\,d\alpha(x)-\sum_{(F,F')\in\mathcal{F}}\int_{rB^n}\mathbf{1}_{A_{F,F'}}(x)V_{n-1}(F\vee(F'+x))\,d\alpha(x)\right|\\&\qquad\le\frac{2}{r^{n+1}}\int_{rB^n}\Big(V_{n-1}(P)+V_{n-1}(P')\Big)\,d\alpha(x)\le\frac{c_1}{r}\end{aligned}\tag{10}$$

for all $r>0$, where here and below $c_1,c_2,\dots$ are some positive numbers depending only on a and α. It follows from Lemma 1, Eq. (8), (e), and (f) that

$$\begin{aligned}&\frac{n-1}{2}\int_0^1 S(sP+(1-s)P'[n-2],Z,\omega)\,ds\\&=\frac{n-1}{2(n+1)}\int_0^1 S(sP+(1-s)P'[n-2],Z',\omega)\,ds\\&=\frac{1}{n+1}\int_{S^{n-1}}\int_0^1 S_{n-2}^{v^\perp}((sP+(1-s)P')|v^\perp,\omega\cap v^\perp)f(v)\,dsd\sigma(v)\\&=\frac{1}{n+1}\int_{S^{n-1}}\int_0^1\sum_{(F,F')\in\mathcal{F}}\mathbf{1}_{B_{F,F'}}(v)V_{n-2}((sF+(1-s)F')|v^\perp)f(v)\,dsd\sigma(v).\end{aligned}\tag{11}$$

For each pair $(F,F')\in\mathcal{F}$ we can choose a vector $y_{F,F'}\in 2aB^n$ such that $\dim\operatorname{aff}(F\cup(F'-y_{F,F'}))=n-2$. As in the proof of Theorem 2, we have for all $x\neq 0$

$$V_{n-1}(F\vee(F'-y_{F,F'}+x))=V_{n-1}(F|x^\perp\vee(((F'-y_{F,F'})|x^\perp)+x)).$$

We thus get from (e) for all $(F,F')\in\mathcal{F}$ and all $r>0$

$$
\begin{aligned}
&\int\limits_{rB^n-y_{F,F'}} \mathbf{1}_{A_{F,F'}}(x)V_{n-1}(F\vee(F'+x))\,d\alpha(x)\\
=&\int\limits_{rB^n}\mathbf{1}_{A_{F,F'}}(x-y_{F,F'})V_{n-1}(F\vee(F'-y_{F,F'}+x))f(x-y_{F,F'})\,d\lambda^n(x)\\
=&\int\limits_0^r\int\limits_{S^{n-1}} t^{n-1}\mathbf{1}_{A_{F,F'}}(tv-y_{F,F'})V_{n-1}(F\vee(F'-y_{F,F'}+tv))f(tv-y_{F,F'})\,d\sigma(v)dt\\
=&\int\limits_0^r\int\limits_{S^{n-1}} t^{n-1}\mathbf{1}_{B_{F,F'}}(v)V_{n-1}(F|v^\perp\vee(((F'-y_{F,F'})|v^\perp)+tv))f(tv-y_{F,F'})\,d\sigma(v)dt\\
=&\int\limits_0^r\int\limits_{S^{n-1}} t^{n-1}\mathbf{1}_{B_{F,F'}}(v)\int\limits_0^t V_{n-2}\Big(\Big(\frac{s}{t}F+\frac{t-s}{t}(F'-y_{F,F'})\Big)\Big|v^\perp\Big)\,ds\,f(tv-y_{F,F'})\,d\sigma(v)dt\\
=&\int\limits_0^r\int\limits_{S^{n-1}} t^{n}\mathbf{1}_{B_{F,F'}}(v)\int\limits_0^1 V_{n-2}((sF+(1-s)F')|v^\perp)\,ds\,f(tv-y_{F,F'})\,d\sigma(v)dt. \qquad (12)
\end{aligned}
$$

Since the symmetric difference $rB^n\triangle(rB^n-y_{F,F'})$ is contained in $R:=((r+2a)B^n\backslash(r-2a)B^n)-y_{F,F'}$ for all $r\ge 2a$, we can use (12) and (11) for $f\equiv 1$ to obtain

$$
\begin{aligned}
&\frac{1}{r^{n+1}}\sum_{(F,F')\in\mathcal{F}}\Bigg|\int\limits_{rB^n}\mathbf{1}_{A_{F,F'}}(x)V_{n-1}(F\vee(F'+x))\,d\alpha(x)\\
&\qquad\qquad -\int\limits_{rB^n-y_{F,F'}}\mathbf{1}_{A_{F,F'}}(x)V_{n-1}(F\vee(F'+x))\,d\alpha(x)\Bigg|\\
\le\ &\frac{1}{r^{n+1}}\sum_{(F,F')\in\mathcal{F}}\int\limits_R\mathbf{1}_{A_{F,F'}}(x)V_{n-1}(F\vee(F'+x))\,d\alpha(x)\\
=\ &\frac{1}{r^{n+1}}\sum_{(F,F')\in\mathcal{F}}\int\limits_{r-2a}^{r+2a}\int\limits_{S^{n-1}}t^n\mathbf{1}_{B_{F,F'}}(v)\\
&\qquad\times\int\limits_0^1 V_{n-2}((sF+(1-s)F')|v^\perp)\,ds\,f(tv-y_{F,F'})\,d\sigma(v)dt\\
\le\ &\frac{\|f\|_\infty}{r^{n+1}}\int\limits_{r-2a}^{r+2a}t^n\,dt\ (n-1)\kappa_{n-1}\int\limits_0^1 S(sP+(1-s)P'[n-2],B^n,S^{n-1})\,ds\\
\le\ &\frac{\|f\|_\infty}{r^{n+1}}\int\limits_{r-2a}^{r+2a}t^n\,dt\ (n-1)\kappa_{n-1}nV(2aB^n[n-2],B^n[2])\ \le\ \frac{c_2}{r} \qquad (13)
\end{aligned}
$$

for all $r\ge 2a$. Since f is uniformly continuous on S^{n-1} and positively homogeneous of degree 0, we deduce from $\|y_{F,F'}\|\le 2a$ for all $(F,F')\in\mathcal{F}$ that for all $\epsilon'>0$ there is a $t_0>0$ such that

$$|f(tv-y_{F,F'})-f(v)|\le\epsilon'$$

for all $v \in S^{n-1}, (F, F') \in \mathcal{F}$ and all $t \geq t_0$. Now (10) – (13) and the triangle inequality imply that

$$\begin{aligned}
&\left| \frac{1}{r^{n+1}} \int\limits_{rB^n} S_{n-1}(P \vee (P'+x), \omega)\, d\alpha(x) - \frac{n-1}{2} \int\limits_0^1 S(sP + (1-s)P'[n-2], Z, \omega)\, ds \right| \\
&\leq \quad \frac{c_1 + c_2}{r} + \frac{1}{r^{n+1}} \sum_{(F,F')\in\mathcal{F}} \int\limits_0^r \int\limits_{S^{n-1}} t^n \mathbf{1}_{B_{F,F'}}(v) \int\limits_0^1 V_{n-2}((sF + (1-s)F')|v^\perp)\, ds \;\times \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times\, |f(tv - y_{F,F'}) - f(v)|\, d\sigma(v) dt \\
&\leq \quad \frac{c_1 + c_2}{r} + \frac{c_3 t_0^{n+1}}{r^{n+1}} + c_4 \epsilon'
\end{aligned}$$

for all $r \geq 2a$, where $t_0 > 0$ is chosen according to $\epsilon' > 0$. If we let ϵ' be small enough, we can find an $r_0 \geq 2a$ so that the above sum does not exceed ϵ for all $r \geq r_0$. Thus (9) is proven and Theorem 4 is established. ■

Proof of Theorem 5. Theorem 5 is deduced from Theorem 4 in the same way as Theorem 3 was derived from Theorem 2. This time we have to use the rotation sum formula

$$\begin{aligned}
&\int\limits_{SO_n} S(K[i], \vartheta K'[j-i-1], K_{j+1}, \ldots, K_{n-1}, Z, \omega \cap \vartheta\omega')\, d\nu(\vartheta) \\
&\quad = \quad \frac{1}{n\kappa_n} S(K[i], B^n[j-i-1], K_{j+1}, \ldots, K_{n-1}, Z, \omega)\; S_{j-i-1}(K', \omega'),
\end{aligned}$$

$i \in \{0, \ldots, j-1\}$, see Schneider [7], p. 295, Eq. (5.3.28). ■

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